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THE ELEMENTS OF MECHANICS

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THE ELEMENTS OF MECHANICS

A TEXT-BOOK FOR COLLEGES
AND TECHNICAL SCHOOLS

BY

W. S. FRANKLIN AND BARRY MACNUTT

New York
THE MACMILLAN COMPANY
LONDON: MACMILLAN & CO., LTD.
1908

Set up, electrotyped and published June, 1907
Reprinted November, 1907; August, 1908

PRESS OF
THE NEW ERA PRINTING COMPANY
LANCASTER, PA.

PREFACE.

In attempting to explain the object which we have had in view in preparing this elementary treatise on Mechanics, we will assume that those to whom this preface is addressed have read our introductory chapter and, particularly, the section entitled The Science of Physics.

We consider that the most important function of the teacher of physics is to build the logical and mechanical structure of the science; the logical structure mainly by lecture and recitation work, including a great deal of practice by the student in numerical calculation, and the mechanical structure of the science mainly by laboratory work. We believe that these two aspects of physics study should run along together. This text, however, is intended as a basis for the work of the class room.

One difficulty in the teaching of physics is that the native sense of most men is incapable, without stimulation, of supplying the materials upon which the logical structure of the science is intended to operate. Those modern studies in psychology which have brought to light the so-called marginal regions of the mind have surely an important bearing upon this question. All elemental knowledge such as the knowing how to throw a ball, how to ride a bicycle, how to swim, or how to use a tool, seems to be locked in these marginal regions as a very substantial but very highly specialized kind of intuition, and the problem of teaching elementary mechanics is perhaps the problem of how, by suggestion or otherwise, to drag this material into the field of consciousness where it may be transformed into a generalized logical structure having traffic relations with every department of the mind. A formal and abstract treatment of the principles of elementary mechanics tends more than anything else to inhibit the influx of this elemental knowledge from the marginal regions of

the mind into the field of consciousness, and results in the building up of a theoretical structure, which can have no effectual traffic with any mental field beyond its own narrow boundaries. Such a state of mind is nothing but a kind of idiocy, and to call it a knowledge of mechanics is silly scholasticism.

Another difficulty is that the human mind, intuitively habituated as it is to consider the immediate practical affairs of life, is hardly to be turned to that minute consideration of apparently insignificant details which is so necessary in the scientific handling even of the most distinctly practical problems.

A third difficulty, which indeed runs through the entire front-of-progress of the human understanding, is that mind-stuff, which has been developed as correlative to certain aspects of our ancestral environment, must be rehabilitated in entirely new relations in fitting a man for the conditions of civilized life. Every teacher knows how much coercion is required for so little of this rehabilitation; but the bare possibility of the process is a remarkable fact, and that it is possible to the extent of bringing a Newton or a Pasteur out of a hunting and fishing ancestry is indeed wonderful.*

Every one is of course familiar with the life history of the butterfly, how it lives first as a caterpillar and then undergoes a complete transformation into a winged insect. It is of course evident that the bodily organs of the caterpillar are not at all suited to the needs of a butterfly, the very food (of those species which take food) being entirely different. As a matter of fact, almost every portion of the bodily structure of the butterfly is dissolved, as it were, into a formless pulp at the beginning of the transformation, and the organization into a flying insect then grows out from a central nucleus very much as a chicken grows in the food-stuff of an egg. So it is in the growth of the civilized man. In early childhood, if the individual is favored by

*Let not the reader suppose that this quality of detachment, which pervades every branch of science, and most of all the science of physics, confronts the specialist only; it is a universal quality and it stands as the most serious obstacle that young men meet with in their study of the elementary sciences.

Fortune, he exercises and develops, more or less extensively, the primitive instincts of the race in a free outdoor life, and the experiences so gained are so much mind-stuff to be dissolved and transformed under more or less coercion and constraint into an effective mind of the twentieth century type.

A fourth difficulty is that the possibility of rehabilitation of mind-stuff has grown up as a human faculty almost solely on the basis of language, and it expresses itself almost entirely in the formation of ideas, whereas a great deal of our knowledge of physics is correlated in mechanisms and can scarcely be reduced to verbal forms of expression. An appreciative friend of the senior author, upon hearing of the proposed elementary treatise on physics, by Nichols and Franklin in 1894, expressed his gratification at what seemed to him to be the promise of a greatly to be desired development of the whole subject of elementary physics out of the idea of energy! But we cannot get away from machinery,* and the mechanisms of manipulation and measurement are essential parts of the structure of the physical sciences without which, the whole thing degenerates into an ineffective philosophy of weak speculation.

The best way to meet this quadruply difficult situation is to relate the teaching of the physical sciences as much as possible to the immediately practical things of life, and, repudiating the repugnance which most of us feel for "literary style" in the handling of what we proudly call the "exact sciences," go in for suggestiveness as the only way to avoid the total inhibition of the little sense that is born with our students. It is *not*, however, a question of exactitude versus suggestiveness, for both are necessary, the one relating chiefly to the realm of ideas and the other to the realms of objective reality.†

* See Art. 134, Chapter XI.

† This statement is purposely left somewhat faulty in view of prevalent ideas as to the unity of the mind. Suggestiveness is the one form of appeal to the marginal contents of the mind, and, except insofar as these marginal contents are elemental in character, suggestiveness relates no more distinctly to objective reality than do the precise ideas which operate almost wholly within the field of consciousness.

Such a method is certainly calculated to 'limber up our theories and put them all at work,' the pragmatic ‡ method our friends the philosophers call it, 'a method which pretends to a conquering destiny'; and whatever one may think of that philosophy of life which exhibits itself in a temperament of the most intensely practical and matter-of-fact type, it is certain that pragmatism is the only philosophy a science teacher can entertain and escape what to him is the most dangerous form of idolatry, science for its own sake.

We wish to place on record our contempt for the poundal as an actual practical unit of force, and also for the "slug" (32.2 pounds) as an actual practical unit of mass. Engineers will perhaps always measure breaking strengths and the like in pounds-weight, and hungry people will perhaps always buy bread and meat by the pound. But most certainly we do need the word *poundal* or the word *slug* so as to enable systematic units and practical units to be connected in intelligible argument. We prefer the word *poundal* for this purpose. There is no doubt in our minds that practical units should be accepted in their entirety, pounds of cargo and pounds-weight of propelling force, and we consider that to attempt to "simplify" the equations of dynamics by the introduction of an unfamiliar and unused unit of force or mass is ridiculous.

In reading over the advanced proofs of the earlier chapters of this book, the authors become painfully aware of the fact that some fifty pages of introductory matter precede what is ordinarily considered to be the proper beginning of the study of elementary mechanics; but surely no one can question the necessity of Chapter II on Physical Measurement, and if anyone, in using the book, wishes to omit Chapter III on Physical Arithmetic, let him do so; we cannot. For the succeeding chapters we offer no apology, indeed, we are inclined in an opposite direction, even to

‡ From the Greek word πραγμα, meaning action, from which our words 'practice' and 'practical' come.

the extent of emphasizing the importance of the chapter on Wave Motion and Oscillatory Motion.

The authors' acknowledgments are due to Professor J. F. Klein for many helpful suggestions, and to Mr. J. H. Wily for the care and promptness with which the illustrations have been prepared.

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SOUTH BETHLEHEM, PA.
April 22, 1907.

SYMBOLS.

(This list includes only those symbols which are often used.)

a, A, linear acceleration.

α, angular acceleration.

B, bulk modulus of a substance.

β, longitudinal strain.

d, D, diameter, or distance, or density.

E, stretch modulus of a substance.

F, force.

g, acceleration of gravity.

k, κ, a proportionality factor.

K, moment of inertia.

l, L, length.

M, m, mass.

μ, coefficient of friction.

n, number of revolutions per second, or number of vibrations per second. Also used for slide modulus of a substance.

p, hydrostatic pressure.

P, power, or longitudinal stress.

ϕ, angle of shear.

r, R, radius.

S, shearing strain.

t, time.

T, torque.

τ, period of one oscillation of a vibrating body.

v, V, linear velocity, or volume.

W, work or energy. Also used for weight.

x, X, an abscissa, or a horizontal distance from a fixed point.

y, Y, an ordinate, or a vertical distance from a fixed point.

ω, angular velocity.

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THE ELEMENTS OF MECHANICS.

CHAPTER I.

INTRODUCTION.

THE LAWS OF MOTION. PHYSICAL MEASUREMENT. THE STUDY OF PHYSICS.

At the beginning of this study of mechanics it is desirable to establish a point of view from which the student must inevitably proceed to a minute consideration of the bare principles of the science, and in attempting to accomplish this feat of insinuation the senior author begs leave to address the student directly as "my young friend," but not, indeed, without some sense of the ludicrous element that lurks in what would seem to be an evident presumption, for the science of mechanics is not supposed to be a place of pleasant paths!

Perhaps every student of engineering realizes the fact that his necessarily specialized course of study is apt to lead him to a rather narrow view of human life as a whole. Every person who studies to work, and few indeed are those who should not, is sure to fall short in this respect. It is important, however, that every engineering student should realize two things in connection with his study of the physical sciences. The first is that the study of the physical sciences is exacting beyond all compromise, involving as it does a degree of coercion and constraint which it is beyond the power of any teacher greatly to mitigate; and the second is

that the completest science stands abashed before the infinitely complicated and fluid array of phenomena of the material world, except only in the assurance which its method gives.

The method of science; that, my young friend, is where constraint and exactions lie, and what I would say to you concerning the change in your mode of thought which this constraint alone can bring about, and which must take place before you can enter into practical life, may best be expressed by referring to the life history of a remarkable little animal, the axolotl, a kind of salamander that lives a tadpole-like youth and never changes to the adult form unless a stress of dry weather annihilates his watery world; but he lives always, and reproduces his kind as a tadpole; and a very odd looking tadpole he is with his lungs hanging from the sides of his head as two large feathery tassels, brownish in tint from the red blood inside and the mud that settles on the outside of the tiny tube-like streamers. When the aquatic home of the axolotl dries up, however, he quickly develops a pair of internal lungs, lops off his tassels, and embarks on a new mode of life on land. If you, my young friend, are to develop beyond the tadpole stage, you must meet, with quick and responsive inward growth, that new and increasing stress of dryness, as many men are wont to call our modern age of science and industry.

Science, as young people study it, has two chief aspects, or, in other words, it may be roughly divided into two parts, namely, the study of those things which *come upon us*, as it were, and the study of those things which we *deliberately devise*. The things that come upon us include weather phenomena and every aspect and phase of the natural world, the things we cannot escape; and the things we devise relate chiefly to the serious work of the world, the things we laboriously build, and the things we deliberately and patiently seek.

You are of course familiar with the accepted division of science into biology, the science of plants and animals, and physics, the science of machines (what is a telescope, a test tube, or a still,

but a machine?). To a certain extent this corresponds to the division of science on the basis of what comes upon us and what we devise; but, strictly speaking, this division of science into biology and physics has but little to do with the distinction between the things that come upon us and the things we devise; and yet it is necessary for our present insinuating purpose to divide science on the basis of something which stands out sharply and distinctly in the experiences of everyday life, so as to leave in each fraction as much as possible of the vital qualities that give pleasure and exact pains. We will therefore, for the present, divide science on the basis of work and — play, if I may be allowed so to mark the contrast between the pranks of the gods and the labors of men.

The Laws of Motion.

The most prominent aspect of all phenomena is motion. In that realm of nature which is not of man's devising, motion is universal; ripple of brook, and gentle flow of river, ocean's swell, and slow recurrent tide, and over all an incessant wandering waft of wind, infinitely varied. In the other realm of nature, the realm of things devised, motion is no less prominent. Every purpose of our practical life is accomplished by movements of the body and by directed movements of tools and mechanisms, such as the swing of the scythe and flail, and those studied movements of planer and lathe from which are evolved strong arm of steam shovel and deft fingers of the loom, the writing hand of the telegraph and tongue of the telephone, and tireless travelers that have no feet nor back to burden with heavy load.

The laws of motion! You, my young friend, must have in some measure my own youthful view, which, to tell the truth, I have never wholly lost, that there is something absurd in the idea of reducing the more complicated phenomena of nature to any orderly system of mechanical law. For, to speak of motion is no doubt to call to your mind first of all the phenomena that are associated with the excessively complicated, incessantly

changing, turbulent and tumbling motion of wind and water. These phenomena have always had the most insistent appeal to us, they have confronted us everywhere and always, and life is an unending contest with their fortuitous diversity, which rises only too often to irresistible sweeps of destruction in fire and flood, and in calamitous crash of collision and collapse where all things commingle in one dread fluid confusion.

The laws of motion! When I was a boy I read many scientific books with great interest, but I never could quite reconcile myself to a certain arrogant suggestion of completeness and universality. The laws of motion! Truly the science of mechanics is circumscribed in utter repudiation of such universal phrase; and yet, curiously enough, the ideas which constitute the laws of motion have an almost unlimited extent of legitimate range, and these ideas must be possessed with perfect precision if one is to acquire any solid knowledge whatever of the phenomena of motion, especially in the realm of devices which so serviceably accomplish the purposes of the human will, for in this realm, at least, everything is really and adequately correlated in a closely woven fabric of reason.

The necessity of precise ideas. Herein lies the impossibility of compromise and the necessity of coercion and constraint; one must think so and so, there is no other way. And the realm of precise ideas, that is the region I intended to symbolize by the land whereon the little tasseled tadpole, the axolotl, is forced to live by unwelcome stress of weather. I remember as a boy a sharp contest in my own mind between an extremely vivid sense of things physical and the constraining function of precise ideas. This contest is perennial, and it is by no means a onesided contest between mere crudity and refinement, for refinement ignores many things. Indeed, precise ideas not only help to form our sense of the world in which we live, but they inhibit sense as well, and their rigid and unchallenged rule would indeed be a stress of dryness.

The laws of motion. I return again and yet again to my sub-

ject, for indeed I am not to be deterred therefrom by any concession of inadequacy, no, nor by any degree of realization of the vividness of your youthful sense of those things which, to suit my narrow purpose, must be stripped completely bare. It is unfortunate, however, for my purpose that the prevailing type of motion, the flowing of water and the blowing of the wind, is bewilderingly useless as a basis for the establishment of the simple and precise ideas which are called the laws of motion, and which are the most important of the fundamental principles of physics. These ideas have, in fact, grown out of the study of the simple phenomena which are associated with the motion of bodies in bulk, without perceptible change of form, the motion of rigid bodies, so called; and these ideas are limited in their strict application to these simple phenomena.

Before narrowing down the scope of my discussion, however, let me illustrate a very general application of the simplest idea of motion, the idea of velocity. You have, no doubt, an idea of what is meant by the velocity of the wind; and a sailor, having what he calls a ten-knot wind, knows that he can manage his boat with a certain spread of canvas and that he can accomplish a certain portion of his voyage in a given time; but an experienced sailor, although he speaks glibly of a ten-knot wind, belies his speech by taking wise precaution against every conceivable emergency. He knows that a ten-knot wind is by no means a sure or a simple thing, with its incessant blasts and whirls; and a sensitive anemometer, having more regard for minutiae than any sailor, usually registers in every wind a number of almost complete but excessively irregular stops and starts every minute and variations of direction that sweep round half the horizon!

We must, as you see, direct our attention to something simpler than the wind. Let us therefore consider the drawing of a wagon or the propulsion of a boat. It is a familiar experience that effort is required to start a body moving, and that continued effort is required to maintain the motion. Certain very simple facts as to the nature of this effort, as to the amount of effort re-

quired to produce motion, and as to the conditions which determine the amount of effort required to keep a body in motion were discovered by Sir Isaac Newton and these facts are called the laws of motion.

The effort required to start a body or to keep it moving is called force. Thus if I start a box sliding along a table I am said to exert a force on the box. I might accomplish the same effect by interposing a stick between my hand and the box, in which case I would exert a force on the stick and the stick in its turn would exert a force on the box. We thus arrive at the notion of force action between inanimate bodies, between the stick and the box in this case, and Newton pointed out that the force action between two bodies A and B always consists of two equal and opposite forces. That is to say, if body A exerts a force on B, then B exerts an equal and opposite force on A, or, to use Newton's words, ACTION IS EQUAL TO REACTION AND IN A CONTRARY DIRECTION.

I might have led up to a statement of this fact by considering the force with which I push on the box and the equal and opposite force with which the box pushes back on me, but if I do not wish to introduce the stick as an intermediary, it is much better to speak of the force with which my hand pushes on the box and the equal and opposite force with which the box pushes back on my hand, because in discussing physical things it is of the utmost importance to eliminate the personal element. I do not think I shall find a better opportunity to explain to you further what I mean by the reference I have made to the curious life history of the little Mexican salamander, the axolotl. It is that our modern industrial life, in bringing men face to face with an entirely unprecedented array of intricate mechanical and physical problems, demands of every one a great and increasing amount of dry, impersonal thinking, and that the precise and rigorous modes of thought of the modern physical sciences are being forced upon widening circles of men with a relentless insistence which must soon give them complete and universal dominion in

those realms of thought which have to do with the harder and more exacting aspects of man's relation to physical things.

When we examine into the conditions under which a body starts to move and the conditions under which a body once started is kept in motion, we shall come across a very remarkable fact, if we are careful to consider every force which acts upon the body, and this remarkable fact is that the forces which act upon a body which remains at rest are related to each other in precisely the same way as the forces which act upon a body which continues to move steadily along a straight path. Therefore, it is convenient to consider, first the relation between the forces which act upon a body at rest, or upon a body in uniform motion, and then to consider the relation between the forces which act upon a body which is starting or stopping or changing the direction of its motion.

Suppose you were to hold a box in mid-air. To do so it would of course be necessary for you to push up on the box so as to balance the downward pull of the earth, the weight of the box, as it is called. Then if I were to take hold of the box and pull upon it in any direction, you would have to exert an equal pull on the box in the opposite direction to keep it stationary. That is to say, the forces which act upon a stationary body are always balanced.

Every one, perhaps, realizes that what I have said about the balanced relation of the forces which act upon a stationary box, is equally true of the forces which act on a box similarly held in a steadily moving railway car or boat. Therefore, the forces which act upon a body which moves steadily along a straight path are balanced.

This is evidently true, as I have pointed out, when the moving body is surrounded on all sides by things which are moving along with it, as a car or a boat; but how about a body which moves steadily in a straight path but which is surrounded by bodies which do not move along with it? You know that some active agent such as a horse or a steam engine must pull steadily

upon such a body to keep it in motion, and you know that if left to itself such a moving body quickly comes to rest. No doubt you have reached this further inference from your experience that this tendency of moving bodies to come to rest is due to the dragging forces, or friction, with which surrounding bodies act upon a body in motion. Thus a moving boat is brought to rest by the drag of the water when the propelling force ceases to act; a train of cars is brought to rest because of the drag due to friction when the pull of the locomotive ceases; a box which is moved across a table quickly comes to rest when left to itself, because of the drag due to friction between the box and the table.

We must, therefore, always consider two distinct forces when we are concerned with a body which is kept in motion, namely, the propelling force due to some active agent such as a horse or an engine, and the dragging force due to surrounding bodies. Newton pointed out that when a body is moving steadily along a straight path, the propelling force is always equal and opposite to the dragging force. Therefore, THE FORCES WHICH ACT UPON A BODY WHICH IS STATIONARY, OR WHICH IS MOVING UNIFORMLY ALONG A STRAIGHT PATH, ARE BALANCED FORCES.

I imagine that you will hesitate to accept as a fact the complete and exact balance of propelling and dragging forces on a body which is moving steadily along a straight path in the open. All I can say is that direct experiment shows it to be true, and that the most elaborate calculations and inferences based upon this notion of the complete balance of propelling and dragging forces on a body in uniform motion are verified by experiment. You may ask, why should a canal boat, for example, continue to move if the pull of the horse does not exceed the drag of the water; but why should it stop if the drag does not exceed the pull? You understand that we are not considering the starting of the boat. The fact is that the conscious effort which one must exert even to drive a horse, the cost of the horse, and the expense of his keep, are what most people think of, however hard one tries to direct their attention solely to the state of tension in

the rope that hitches the horse to the boat after the boat is in full motion; and most people come upon the idea that if the function of the horse is simply to balance the drag of the water so as to keep the boat from stopping, then why should there not be some way to avoid the cost of so insignificant an operation? There is, indeed, an extremely important matter involved here which we will consider when we come to the discussion of work and energy; but it has no bearing on the matter of the balance of propulsion and drag on a body which moves steadily along a straight path.

Let us now consider the relation between the forces which act upon a body which is changing its speed, upon a body which is being started or stopped, for example. I suppose that you have noticed how a horse strains at his rope when starting a canal boat, especially if the boat is heavily loaded, and how the boat continues to move for a long time after the horse ceases to pull. In the first case, the pull of the horse greatly exceeds the drag of the water, and the speed of the boat increases; and in the second case, the drag of the water of course exceeds the pull of the horse, for the horse is not pulling at all, and the speed of the boat decreases. When the speed of a body is changing, the forces which act on the body are unbalanced, and we may conclude that *the effect of an unbalanced force acting on a body is to change the velocity of the body;* and it is evident that the longer the unbalanced force continues to act the greater the change of velocity. Thus if the horse ceases to pull on a canal boat for one second the velocity of the boat will be but slightly reduced by the unbalanced drag of the water, whereas if the horse ceases to pull for two seconds the decrease of velocity will be much greater. *In fact the change of velocity due to a given unbalanced force is proportional to the time that the force continues to act.* This is exemplified by a body falling under the action of the unbalanced pull of the earth; after one second it will have gained a certain amount of velocity (about 32 feet per second), after two seconds it will have made a total gain of twice as much velocity (about 64 feet per second), and so on. Furthermore, since the velocity produced by an unbalanced

force is proportional to the time that the force continues to act, it is evident that the effect of the force should be specified as so much velocity produced per second, exactly as in the case of earning money, the amount one earns is proportional to the length of time that one continues to work, and we always specify one's earning capacity as so much money earned per day.

Everyone knows what it means to give an easy pull or a hard pull on a body. That is to say, we all have the ideas of greater and less as applied to forces. Everybody knows also that if a horse pulls hard at a canal boat, the boat will get under way more quickly than if the pull is easy, that is, the boat will gain more velocity per unit of time under the action of a hard pull than under the action of an easy pull. Therefore, any precise statement of the effect of an unbalanced force on a given body must correlate the precise value of the force and the exact amount of velocity produced per unit of time by the force. This seems a very difficult thing, but its apparent difficulty is very largely due to the fact that as yet we have not agreed as to what we are to understand by the statement that one force is precisely three, or four, or any number of times as great as another. Suppose, therefore, that we agree to call one force twice as large as another when it will produce in a given body twice as much velocity in a given time (remembering of course that we are now talking about unbalanced forces, or that we are assuming for the sake of simplicity of statement, that no dragging forces exist). As a result of this definition we may state that the amount of velocity produced per second in a given body by an unbalanced force is proportional to the force.

Of course we know no more about the matter in hand than we did before we adopted the definition, but we do have a good illustration of how important a part is played in the study of science, by what we may call making up one's mind, in the sense of putting one's mind in order. This kind of thing is very prominent in the study of elementary physics, and by that rather indefinite eference, in my story of the little tasseled tadpole, to an inward

growth so needful to you before you can hope for any measure of success in our modern world of scientific industry, I meant to refer to this thing, the "making-up" of one's mind. Nothing is so essential in the acquirement of exact and solid knowledge as the possession of precise ideas, not indeed that a perfect precision is necessary as a means for retaining knowledge, but *that nothing else so effectually opens the mind for the perception even of the simplest evidences of a subject.** We may illustrate these things further by following up our discussion of the laws of motion.

We have now settled the question as to the effect of different unbalanced forces on a given body on the basis of a very general experience, and by an agreement as to the precise meaning to be attached to the statement that one force is so many times as great as another; but how about the effect of a given force upon different bodies, and how may we identify the force so as to be sure that it is the same? As to the identification, a given force may be made to act on any body by causing a given body to exert the force, and by considering whether the *reaction* produces the same effect on the given body in each case. Thus a spring dynamometer may be used to exert a given force on any body, the reaction on the spring dynamometer causing a given stretch of the spring. As to the effect of a given unbalanced force in producing velocity of different bodies, three things have to be settled by experiment.

(*a*) In the first place let us suppose that a certain force A is twice as large as a certain other force B, according to our agreement, because the force A produces twice as much velocity every second as force B when the one and then the other of these forces is caused to act upon a given body, a piece of lead, for example. Then, will the force A produce twice as much velocity every second as the force B whatever the nature and size of the body, whether it be wood, or ice, or sugar? Experiment shows that it will.

*Opens the mind, that is, for those things which are conformable to or consistent with the ideas. The history of science presents many cases in which accepted ideas have closed the mind to contrary evidences for many generations. Let young men beware!

(*b*) In the second place, suppose that we have such amounts of lead, of iron, of wood, etc., that a certain given force produces the same amount of velocity per second when it is made to act, as an unbalanced force, upon one or another of these various bodies. Then what is the relation between the amounts of these various substances? Experiment shows that they all have the same mass in grams, or pounds, as determined by a balance. That is, a given force produces the same amount of velocity per second in a given number of grams of any kind of substance. Thus the earth pulls with a certain definite force (in a given locality) upon M grams of any substance and, aside from the dragging forces due to air friction, all kinds of bodies gain the same amount of velocity per second when they fall under action of the unbalanced pull of the earth.

(*c*) In the third place, what is the relation between the velocity per second produced by a given force and the mass in grams of the body upon which it acts? Experiment shows that the velocity per second produced by a given force is inversely proportional to the mass of the body upon which the force acts.

The effect of an unbalanced force in producing velocity may therefore be summed up as follows: THE VELOCITY PER SECOND PRODUCED BY AN UNBALANCED FORCE IS PROPORTIONAL TO THE FORCE AND INVERSELY PROPORTIONAL TO THE MASS OF THE BODY UPON WHICH THE FORCE ACTS. Furthermore, THE VELOCITY PRODUCED BY AN UNBALANCED FORCE IS ALWAYS IN THE DIRECTION OF THE FORCE.

PHYSICAL MEASUREMENT.

Among primitive races all things subject to exchange or barter are estimated by simple counting. Thus a Tartar herdsman estimates his wealth by counting his cattle. With the growth of civilization, however, there has been a great increase in the variety of useful and exchangeable commodities, and many of these commodities, molasses, for example, cannot be estimated by simple counting. The result has been that the simple operaion of counting, which, of course, can be applied only to groups

of separate and distinct things, has developed into the operation called measurement, in which a continuous whole is estimated numerically by dividing it into equal, unit parts, and counting these parts. Thus oil or wine is counted out by means of a gallon measure, and cloth is counted out by means of a yard-stick.

In many kinds of measurement, the two distinct operations, (*a*) *dividing into equal unit parts* and (*b*) *counting* are obscured by the use of more or less elaborate measuring devices, but every measurement of whatever kind does, in fact, consist of these two fundamental operations. Thus in measuring a length by means of a *scale of inches*, the operation of dividing into unit parts has been performed once for all by the maker of the scale, and in this case the operation of counting is, in large part, "ready-made" by the numbers stamped on the scale. In the weighing of a consignment of coal or iron, the operation of dividing into unit parts has been performed once for all by the maker of the *set of weights* and of the *divided balance beam*, and the operation of counting is, in large part, "ready-made" by the numbers stamped upon the weights and upon the beam.

The long experience of the race in estimating by the simple counting of separate things has given rise to a sense of sharp distinction between any two numbers, thus 1000 horses is clearly not the same thing as 999 horses; *but this sharp distinction between approximately equal numbers is devoid of physical significance in the case of numbers derived by measurement, because of the approximate character of the operation of dividing a whole into unit parts.* A person might buy a herd of horses supposing the number to be 1000, whereas a correct count would show 999; and although the purchaser might reasonably say, "Oh, let it go it makes no difference," still the fact would remain that 999 horses is not 1000 horses; but suppose a man were to buy 1000 yards of cloth, he might remeasure the cloth and count 999 yards, but in remeasuring the day may have been damp, or he may not have stretched the cloth in the same way as the manufacturer, or he may have taken more or less pains in fitting the

yard-stick to the successive portions of the cloth, or his yard-stick may have been in error. The fact is that it is impossible to show that 1000 yards of cloth *is not* 999 yards of cloth, except by reasoning that 1000 pieces of silver is not 999 pieces of silver. The difficulty is that a yard of cloth is not a separate thing whereas a piece of silver is.

Nothing is more amusingly indicative of a disregard of physical facts than to see a long array of digits carried laboriously through an arithmetical calculation which is based upon numerical data obtained by physical measurement. For example, a man weighs a body in air and then suspends it by a string under water and weighs it again, obtaining 105.26 grams and 74.63 grams respectively; and from these data he calculates the specific gravity to be 3.436500489 +. This would be sufficiently amusing if it were certain that the two numbers 105.26 and 74.63 were free, as far as they go, from every influence like those described above as modifying the measured length of a piece of cloth; but if this is not certain, then the thing is indeed ridiculous.

The operation of dividing a length or an angle into equal unit parts for the purpose of measurement, is an operation of *fitting* a standard to each part, an operation of congruence; and the actual measurement of any physical whole — let us not speak of it as a quantity until we have attached a number to it — depends upon one or another variety of congruence as a basis for the assumption of equality of the parts which are to be counted. Thus a pendulum may be assumed to mark off equal intervals of time because each movement of the pendulum is like the one that follows; and the equal arm balance is a device for indicating a certain kind of congruence between the body which is being weighed and the combination of weights which balances it.

The fundamental meaning of a physical quantity originates in and is defined by the actual operation of measuring that quantity. Thus it is sheer nonsense to define the mass of a body as "the mount of material the body contains." The mass of a body,

as a quantity, is defined by the operation of weighing by a balance; and, since the result of this operation is always the same, within the limits of error, for a given amount of any substance, it is permissible to use this result as a measure of the amount of the substance. *Nearly every physical definition, rightly understood, is an actual physical operation.*

The Science of Physics.

"We advise all men" says Bacon "to think of the true ends of knowledge, and that they endeavor not after it for curiosity, contention, or the sake of despising others, nor yet for reputation or power or any other such inferior consideration, but solely for the occasions and uses of life." It is, indeed, impossible to imagine any other basis upon which the study of physics can be justified than for the occasions and uses of life. At any rate, more than nine-tenths of the subject matter of physics now relates to the conditions which have been elaborated through the devices of industry, and the study of physics has to do almost wholly with devised phenomena, as exemplified in our mills and factories, in our machinery of transportation, in optical and musical instruments, in the means for the supply of power, heat, light, and water for general and domestic use, and so on.

From this extremely practical point of view it may seem that there can be nothing very exacting in the study of the physical sciences; but what is physics? That is the question. One definition at least we must repudiate; it is not "The science of masses, molecules, and the ether." Bodies have mass and railways have length, and to speak of physics as the "science of masses" is as silly as to define railroading as the "practice of lengths," and nothing as reasonable as this can be said in favor of the conception of physics as the science of molecules and the ether; it is the sickliest possible notion of physics even if a student really gets it, whereas the healthiest notion, even if a student does not wholly grasp it, is that physics is the science of the ways of taking hold of things and pushing them!

Bacon long ago listed in his quaint way the things which seemed to him most needful for the advancement of learning. Among other things he mentioned "A New Engine or a Help to the mind corresponding to Tools for the hand," and the most remarkable aspect of physical science is that aspect in which it constitutes a realization of this New Engine of Bacon. We continually force upon the extremely meager data obtained directly through our senses, an interpretation which, in its complexity and penetration, would seem to be entirely incommensurate with the data themselves, and the possibility of this forced interpretation depends upon the use of two complexes: (*a*) A *logical structure*, that is to say, a body of mathematical and conceptual theory which is brought to bear upon the immediate materials of sense, and (*b*) a *mechanical structure*, that is to say, either (1) a carefully planned arrangement of apparatus, such as is always necessary in making physical measurements, or (2) a carefully planned order of operations, such as the successive operations of solution, reaction, precipitation, filtration, and weighing in chemistry.

These two complexes do indeed constitute a New Engine which helps the mind as tools do help the hand, it is through the enrichment of the materials of sense by the operation of this New Engine that the elaborate interpretations of the physical sciences are made possible, and the study of elementary physics is intended to lead to the realization of this New Engine: (*a*) By the building up in the mind, of the logical structure of the physical sciences; (*b*) by training in the making of measurements and in the performance of ordered operations, and (*c*) by exercises in the application of these things to the actual phenomena of physics and chemistry at every step and all of the time with every possible variation.

That, surely, is a sufficiently uncompromising program. Indeed, many raise the objection that a rigorous presentation of the structure of physics, logical and mechanical, is highly unsatisfactory and uninstructive, and of course this is true if the physical facts themselves are lost to view; but any student who

indulges a fancied interest in the "results" of science, and who, becoming absorbed, for example, in a purely descriptive treatise on recent researches on "light pressure" and the cause of comets' tails, holds his imagination unresponsive to a discussion of velocity and acceleration such as that given in the next chapter, any student who does this, I say, should be treated honestly and placed under the instruction of Jules Verne, where he need not trouble himself about foundations, but may follow his teacher pleasantly on a care-free trip to the moon, or with easy improvidence embark on a voyage of twenty thousand leagues under the sea. There are too many people who fancy that they have an interest in the "results" of science and who, poor fools, invest in Keeley Motors and Sea Gold Companies because, forsooth, the desired result is so clearly evident.

The greatest fault in an elementary treatise on physics is obscurity in that region which lies between raw unformulated nature on the one hand and the highly elaborated ideas and conceptions of physical theory on the other hand. "Our method," says Bacon, "is continually to dwell among things soberly, without abstracting or setting the mind farther from them than makes their images meet," and "the capital precept for the whole undertaking is that the eye of the mind be never taken off from things themselves, but receive their images as they truly are, and God forbid that we should offer the dreams of fancy for a model of the world."

CHAPTER II.

MEASUREMENT OF LENGTH, ANGLE, MASS AND TIME.

1. Units of length. — The *meter* is the distance, at the temperature of melting ice, between two lines on a certain platinum-iridium bar which is preserved in the vaults of the International Bureau of Weights and Measures near Paris. Very accurate copies * of this bar are deposited in Washington and in London, and the legal units of length in all countries are now defined in terms of the meter.

It was intended originally that the meter should be equal to one ten-millionth part of the distance from the equator to the poles of the earth, but copies of the meter can be made with much greater accuracy and with incomparably greater ease by direct comparison with the standard meter bar in Paris than by comparison with the earth's quadrant, and therefore the definition of the meter as a ten-millionth of the earth's quadrant is entirely illusory.

The *yard* is now defined as $\frac{3600}{3937}$ of a meter.†

2. Measurement of length: scale and vernier. — Lengths are commonly measured by means of divided scales. The measurement is accomplished by counting the number of scale divisions included in the length to be measured. Fractions of a division may be estimated by the eye or determined by means of a device called a *vernier*. The general principle of the vernier is as fol-

* These copies are called the *international prototypes* of the meter. See *Nature*, Vol. 51, p. 420, February 28, 1895.

† On April 5, 1893, a decision was reached by the United States Superintendent of Weights and Measures, with the approval of the Secretary of the Treasury, that the meter and the kilogram would be regarded as the fundamental standards not only for metric units but also for the customary units of length and mass. See a History of the Standard Weights and Measures of the United States by Louis A. Fischer, Vol. I., pp. 365–381, *Bulletin of the Bureau of Standards* (United States Department of Commerce and Labor).

lows: The divided scale is represented by S, Fig. 1. Let us call the divisions on this scale millimeters for brevity. A short auxiliary scale V, the vernier, is $(n - 1)$ millimeters long and it is divided into n equal parts. The diagram, Fig. 1, is constructed for $n = 10$. Let the space f be the fraction of a millimeter to be determined, and let it be equal to a/n of a millimeter; the space g is $1/n$ mm. shorter, the space h is $2/n$ mm. shorter, and so on; so that the a^{th} mark on the vernier is coincident with a mark on the scale. Thus a is determined. The number of the mark on the vernier which is coincident with a mark on the scale is the

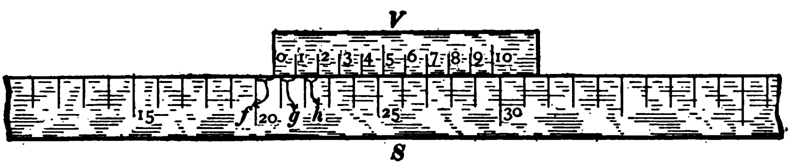


Fig. 1.

numerator and the number of divisions on the vernier is the *denominator* of the fraction which expresses the space f in terms of a scale division. The position of the zero mark of the vernier on the scale is called the *reading of the vernier*. Thus the reading of the vernier in Fig. 1 is 20.7 mm.

To measure a given length, any chosen point on the vernier is made to coincide with one end of the length, and the vernier reading is taken. The vernier is then moved until the same point of the vernier coincides with the other end of the length, and the vernier reading is again taken. The difference of these two vernier readings is the distance the vernier has been moved and it is equal to the given length.

The vernier is frequently used to determine fractions of divisions on divided circles.

3. The dividing engine is a machine for the manufacture of divided scales and for the precise measurement of length. An accurate * horizontal screw, having a divided circular head for

* An interesting description of the method of making an accurate screw is given in the Encyclopedia Britannica, 9th edition, article *screw*.

Fig. 2.

indicating fractions of a turn, engages a nut which pushes a sliding carriage along a track on a heavy metal bed-plate which projects to one side of the carriage as a platform, as shown in Fig. 2. On the carriage are mounted a graving tool and a reading microscope (the dividing engine shown in Fig. 2 is intended only for the making of divided scales, its screw is standardized by the manufacturer, and it is not intended for use in measuring the

length of an object). To standardize the screw a meter bar is placed on the platform, the screw is turned until the microscope sights at one end of the meter, and then the exact number of turns, a, required to move the microscope to the other end of the meter is counted, fractions of turns being estimated by the divided circular head.

To measure the length of any object, the object is placed on the platform, the screw is turned until the microscope sights at one end of the object, and then the exact number of turns, b, required to move the microscope to the other end of the object is counted. The length of the object is then known to be b/a of a meter.

To manufacture a divided scale, a blank bar is placed upon the platform of the dividing engine, the screw is turned until the graving tool is conveniently near to one end of the bar, and a mark is made; the screw is then turned, a/n turns, and another mark is made, and so on, thus dividing the bar into $n$ths of a meter.

4. Units of angle. The angle all the way around a point, that is, the angle which is represented by the entire circumference of a circle, is a natural unit of angle, and it is not necessary to preserve a material standard of the unit angle. The unit of angle which is universally used for purposes of measurement is the *degree*, it is equal to $\frac{1}{360}$ of the angle which is represented by the entire circumference of a circle. In many calculations, however, it is convenient to express an angle as follows: Imagine a circle of radius r drawn with its center at the apex of an angle, and let a be the length of the arc of the circle which is included between the boundaries of the angle; then the ratio a/r has a fixed value for a given angle, and the value of this ratio is frequently used as a numerical measure of the angle. The unit angle in this system is the angle of which the length of the subtending arc is equal to the radius and it is called the *radian*.

Measurement of angle. — In many instruments, angles are measured by means of the divided circle. The divided circle is

placed with its center at the apex of the angle, and the angle is measured by counting the number of circle divisions between the lines which determine the angle. These lines are established by a pair of sights fixed to an arm called an alidade. This alidade rests flat on the divided circle, turns on a pivot at the center of the circle, and carries usually two verniers, one at each end.

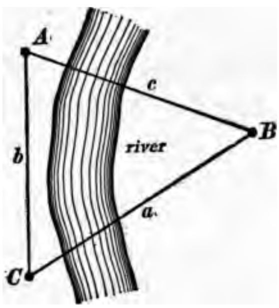


Fig. 3.

Indirect measurement of length and angle: triangulation. — Consider a triangle ABC, Fig. 3, of which the side b, only, is accessible. The lengths of the sides a and c and the angle B may be calculated by trigonometry when the side b and the angles A and C have been measured. This method for determining inaccessible lengths is called *triangulation.* It is much used in surveying and in astronomy.

Poggendorff's method for measuring angle. — In many instruments it is necessary to measure the angle through which a suspended body is turned. For this purpose a small mirror, mm, Fig. 4, is fastened to the suspended body so as to turn with it. A straight scale, ss, is placed at a measured distance, d, in front of and parallel to the mirror. A telescope, which establishes a sight line, is placed so that the scale is seen through it in the mirror,

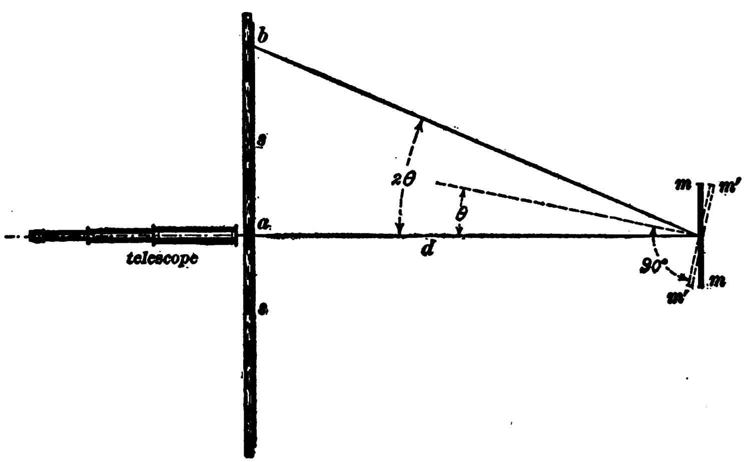


Fig. 4.

the sight line being perpendicular to the scale. The reading, a, of this sight line on the scale is taken; the mirror then turns through the angle θ into the position $m'm'$,

and the scale reading, b, is again taken. The sight line is deflected through the angle 2θ, so that $(a-b)/d = \tan 2\theta$, from which the angle θ may be calculated.

5. Units of area. — The unit of area is defined as the area of a square * of which the side is of unit length.

Measurement of area. — Area is determined fundamentally by calculation from measured linear dimensions.

The *planimeter* is an instrument for measuring irregular plane areas; strictly, for determining the ratio of two such areas, for the instrument is standardized by measuring with it a regular figure of which the area is known by calculation from measured linear dimensions.

Theory of the planimeter. — Consider a line AB, Fig. 5, of length l, moving in any manner in the plane of the paper. The motion of the line may at each instant be considered as compounded † of a motion of translation and a motion of rotation, with angular velocity $d\theta/dt$, about an arbitrary point p distant D from the center of the line. The area swept by this moving line is considered positive when the line sweeps over it from left to right to an observer looking from A towards B. Let v be the resolved part, perpendicutar to the line, of its velocity of translation. The line sweeps over area at the rate lv, because of its motion of translation, and at the rate $lD \cdot d\theta/dt$, because of its motion of rotation, so that the total rate at which the line sweeps area is at each instant:

$$\frac{dA}{dt} = lv + lD\frac{d\theta}{dt} \tag{i}$$

Let a wheel, radius r, mounted at p, with its axis parallel to AB, be allowed to roll on the paper as the line moves, and let $d\psi/dt$ be the angular velocity of rolling of the wheel. Then $v = r \cdot d\psi/dt$, and equation (i) becomes

$$\frac{dA}{dt} = lr\frac{d\psi}{dt} + lD\frac{d\theta}{dt} \tag{ii}$$

or ‡

$$A = lr\psi + lD\theta \tag{iii}$$

in which A is the total area swept by the line during the time that the wheel has turned through the angle ψ and the line has turned about p through the angle θ.

* The *circular mil*, much used by electricians as a unit area, is the area of a circle one *mil* ($\frac{1}{1000}$ inch) in diameter. The area of any circle in circular mils is equal to the square of its diameter in mils.

† See Article 77.

‡ See Article 20.

If the line comes back to its initial position, or parallel thereto, so that θ is equal to zero,* then equation (iii) becomes

$$A = lr\psi \tag{iv}$$

That is, the total area swept by AB is proportional to the angle ψ turned by the rolling wheel, and *the circumference of the wheel may be so divided as to read areas directly.*

Let one end of AB, Fig. 6, be constrained to move along a branch AC of any curve, while the other end passes once around a closed line D. Any area outside of D which is swept over by the line AB at all is swept as many times to the right as

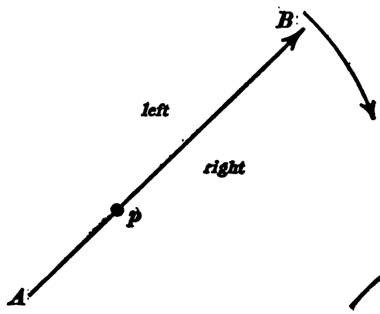


Fig. 5.

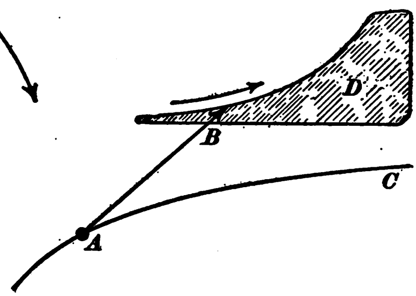


Fig. 6.

to the left, and all parts of D are swept once more to the right than to the left, so that the total area swept by AB is equal to the area of D. In its simplest form the planimeter consists of an arm AB with a rolling wheel. The end A is constrained to move on the arc of a circle, being hinged to one end of an auxiliary arm, the other end of which is fixed by a pivot.

6. Units of volume. — The unit of volume is defined as the volume of a cube of which the edge is of unit length.

Measurement of volume. — Volume is determined fundamentally by calculation from measured linear dimensions. Thus the volume of a rectangular parallelopiped is equal to the product of its length, breadth, and height.

Volumes of liquid and of grain are usually measured by means of a vessel of known volume. The *graduate* is a vessel upon the side of which is a scale from which the volume of a portion of liquid may be read off; it is much used by druggists and

* The line may come back to its initial position so that θ equals $2n\pi$, where n is any whole number.

by chemists. A method for determining volume by weighing is described under density.

7. Mass. — Everyone is familiar with the measurement of material *by volume* and *by weight*, but everyone does not distinguish between the two methods in common use for measuring by weight, namely, the method in which the *spring scale* is used and the method in which the *balance scale* is used. The spring scale measures the force with which the earth pulls a body, and the weight of a given body as indicated by a spring scale is greater or less according as the pull of the earth for the given body is greater or less; the indication of the balance scale, on the other hand, does not vary with the gravity-pull of the earth, inasmuch as the gravity-pull on the weights and the gravity-pull on the weighed body both change together, the indications of a balance scale are therefore independent of the value of gravity. *The result of the operation of weighing by a balance scale is called the mass of a body.**

Considering that weighing is nearly always done by the balance scale, it is evident that what is popularly called the weight of a body is what scientific men call the *mass* of the body, and it is important to remember that the force with which the earth pulls a body is called the *weight* of the body by scientific men. The verb *to weigh* means nearly always the determination of the mass of a body by means of the balance scale.

* We might agree to consider the mass of one body to be twice as great as the mass of another body when a given force will produce half as much velocity per second when it acts upon the first body as it will when it acts upon the second body; but the proper definition of a quantity is the definition which corresponds to the fundamental method which is actually used in measuring that quantity. No one ever thinks of measuring out a ton of coal by loading it upon a "frictionless" car and finding how many times less velocity is imparted to it in a second by a given force than would be imparted to a standard pound by the same force! Compare this as an actual operation with the measuring of coal by the ordinary platform scale, using standard pounds and fractions of a pound as counterpoises! It is all very well to *talk* about defining the mass of a body in accordance with the above utterly impracticable method of measuring its mass, but sensible men always *define* things in physics in the way they *do* them.

If the force with which the earth pulls the unit of mass *at a given place on the earth* is adopted as the unit of force and called the *pound of force* (or the *gram of force*), then the weight of any body in pounds of force *at that place on the earth* will be exactly equal to its mass in pounds, and, in fact, the weight of a body at any other place on the earth will not differ from its mass by more than two or three tenths of one per cent.

Units of mass. The *kilogram* is the mass of a certain piece of platinum which is preserved in the vaults of the International Bureau of Weights and Measures. Very accurate copies * of the kilogram are deposited in Washington and in London, and the legal units of mass in all countries are now defined in terms of the kilogram. Thus the *pound* (avoirdupois) is defined as 1/2.204622 of a kilogram.†

It was intended originally that the kilogram should be equal to the mass of a cubic decimeter (1,000 cubic centimeters) of water at a temperature of 4° C. and at atmospheric pressure, but the extreme difficulty of reproducing accurate copies of the kilogram on the basis of this definition makes the definition not only impracticable but illusory.

Measurement of mass. — The analytical balance consists of a delicately mounted equal-arm lever with pans suspended from its ends. *The balance is used simply for indicating the equality of the masses of two bodies*, that is, two bodies are said to have equal masses when they balance each other when suspended from the ends of an equal-arm lever.

The determination of the mass of a body by means of the balance depends upon the use of a *set of weights* which may be combined in such a way as to match the mass of the body. Such a set of weights may be made by taking two pieces of metal weighing together one kilogram and then making them balance each other by cutting metal off from one and adding the shavings to

* See *Nature*, Vol. 51, p. 420, February 28, 1895.

† See *Bulletin of the Bureau of Standards* (United States Department of Commerce and Labor), Vol. I., pp. 365–381.

the other, thus giving a half-kilogram weight. Then a quarter-kilogram weight may be made in the same way and so on. A set of weights more convenient in use is a set which contains a five, a two, and two ones of each — units, tens, hundreds, etc., of grams.

8. Density. Every one knows that lead is heavier than cork, and every one feels instantly that there is some hitch in the question "Which is heavier, a pound of lead or a pound of cork." The word heaviness has, in fact, a double meaning. A pound of lead is not heavier than a pound of cork, because to specify a pound in each case is to imply that both have been weighed and that the result is the same for each, namely, one pound of lead and one pound of cork; but *lead is heavier than cork in the sense that a piece of lead weighs more than an equal bulk of cork.* The word *density* is used to designate this idea of heaviness as an inherent property of a substance. Thus, lead has greater density than cork.

The density of a substance is its mass per unit of volume, that is, it is the mass of a body divided by the volume of the body. The density of a substance may be specified in terms of any units of mass and volume. Thus the density of a liquid such as oil is usually specified by commercial men as so many pounds per gallon; the density of stone is usually specified by a building contractor as so many pounds per cubic foot, and so on. In this treatise density will be expressed in grams per cubic centimeter.

The *specific gravity* of a substance at a given temperature is the ratio of the density of the substance to the density of water at the same temperature, that is to say, the specific gravity of a substance is its density expressed in terms of the density of water as unity.

Measurement of density. — The density of a substance is determined fundamentally by weighing a measured volume of the substance. Thus the density of water has been very carefully determined at the International Bureau of Weights and Measures * as

* W. Marek, *Trav. et Mém. du Bureau internat. des Poids et Mes.*, III, D 81, 1884.

follows: The volume of an accurately cut glass cube is calculated from its measured linear dimensions and the cube is weighed in air and then suspended under water and weighed again. The difference, duly corrected for buoyancy of air, is the mass of a volume of water equal to the volume of the cube. The density of water at a given temperature being thus determined, its density at other temperatures is found by the method of Regnault which is described in the chapter on thermometry.

Measurement of specific gravity. — The specific gravity of a substance is determined by weighing equal volumes of the substance and of water. The simplest case is in the use of the specific gravity bottle for determining the specific gravity of a liquid. The bottle is weighed empty, then it is weighed when filled with the given liquid, and then it is weighed when filled with water. Other methods for determining specific gravity are discussed in the chapter on hydrostatics. When the specific gravity of a substance has been found at a given temperature, the density of the substance at the given temperature may be found by multiplying the specific gravity of the substance by the known density of water at that temperature. See table *density of water* in the chapter on thermometry.

Gravimetric method for measuring volume. — The volume of a vessel at a given temperature may be determined with great accuracy by weighing the vessel empty, and then weighing it when filled with water or mercury at the given temperature. The density of the water or mercury being known, the volume of the vessel is easily calculated from the net weight (mass) of the water or mercury. The measuring vessels used by chemists are standardized in this way.

9. Time. — The mean solar day is the natural unit of time, and the *second*, which is the accepted unit of time in many physical measurements, is the 86,400th part of a mean solar day.

Measurement of time. — Any movement of a body which repeats itself in equal intervals of time is called *periodic motion;* single movements are called *vibrations.* All methods, with un-

important exceptions, for measuring time depend upon periodic motion. A vibrating *pendulum* is the most familiar example. *The number, a, of vibrations in a day, and the number, b, in the interval to be measured, are counted. The interval is then known to be equal to b/a of a day.* A *clock* is simply a machine for maintaining and counting the vibrations of a pendulum. In portable clocks a *balance wheel* takes the place of a pendulum.

Anything which affects the time of vibration of a pendulum leads to erroneous values for time intervals as measured by a clock. The time of vibration of a pendulum is affected (*a*) by temperature, on account of increase of length of the pendulum with temperature; (*b*) by variations of atmospheric pressure, on account, mainly, of variation of buoyant force of air with pressure; (*c*) by variation in amplitude of vibration; and (*d*) by irregularities in the manner in which impulses are imparted to the pendulum by the clockwork. The influence of temperature is avoided by using what are called *compensated* pendulums, which do not change their effective length with temperature. The variations due to (*c*) are in part obviated by providing constant driving power, which requires the gears and escapement to be of fine workmanship. The variations due to (*d*) are obviated by using what are called *dead beat* escapements, which impart their impulse to the pendulum at the instant when it passes through the vertical position.*

10. The chronograph. — The determination of a time interval by means of a clock requires the clock reading to be taken at the beginning and at the end of the interval. Practice enables an observer to take the clock reading at the instant of a given signal accurately to a tenth of a second, with an approximately constant "personal" error which does not greatly affect the value of the interval. In the practice of this method, which is called the *eye and ear* method, the observer looks for the signal and listens to the beats of the clock.

* For discussion of errors, and description of escapements, see Encyclopedia Britannica, 9th ed., article *Clock*. For description of compensated chronometer balance and chronometer escapement, see Lockyer's book entitled *Star-gazing*, pp. 175 to 210.

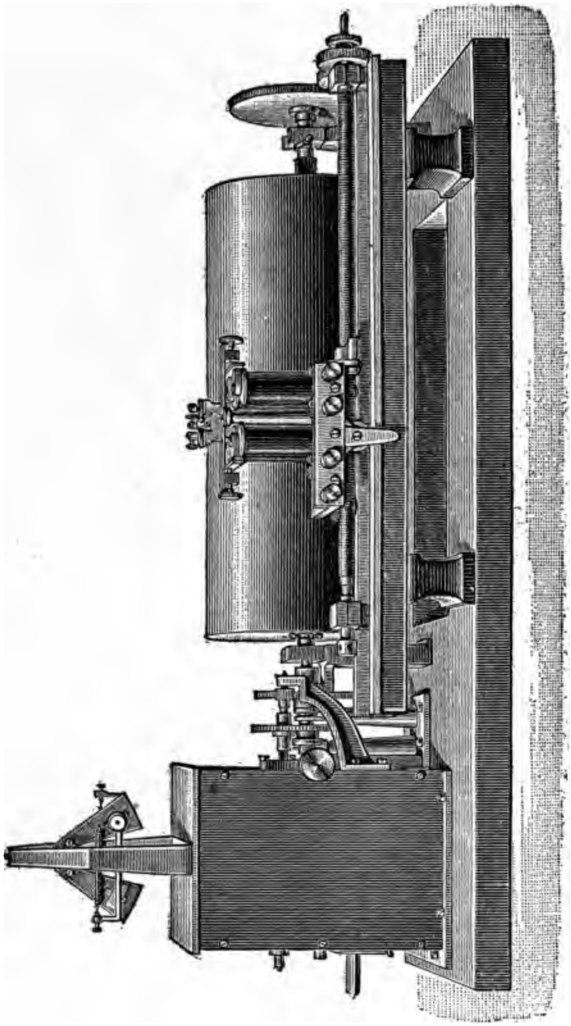


Fig. 7.

The chronograph is an instrument for enabling clock readings to be taken with greater ease and accuracy than is possible by "eye and ear." A pen traces a line upon a uniformly moving strip of paper. This pen is fixed to the armature of an electro-magnet, which is excited at each beat of the clock by an electric current controlled by a contact device actuated by the clock

pendulum. A kink is thus made in the traced line at each beat of the pendulum. At the instant for which the clock reading is desired, the electro-magnet is momentarily excited by pressing a key which closes an auxiliary electric circuit, thus making an extra kink in the traced line, and the clock reading is determined by measuring off the position of this extra kink among the kinks produced by the beats of the pendulum.

The above description applies to the essential features of the chronograph. The form of the instrument as ordinarily used for accurate time observations is shown in Fig. 7. A large sheet of paper is wrapped around a cylinder which is rotated at a uniform speed by clock work. The tracing pen and electro-magnet are mounted on a sliding carriage which is slowly moved parallel to

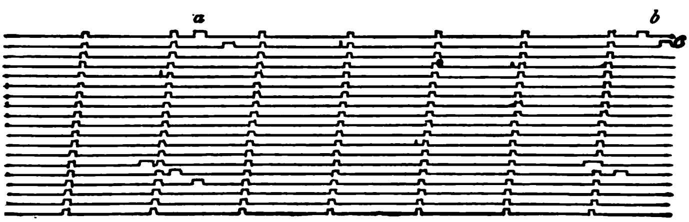


Fig. 8.

the axis of the rotating cylinder by a screw which is driven by the same clock-work that drives the cylinder. The pen thus traces a helical line on the paper-covered cylinder, so that the large sheet of paper is equivalent to a very long narrow strip.

Figure 8 shows a reduced facsimile of a portion of the paper sheet upon which a chronographic record has been made. The regularly spaced kinks are those produced by the beats of the clock, and the kinks marked *a*, *b*, *c*, etc., are those produced by depressing the key.

Problems.

1. A scale has half inch divisions. A vernier to be used with this scale is $7\frac{1}{2}$ inches long and it is divided into sixteen equal parts. To what fractional part of an inch does the vernier read?

2. The divisions on the circle of a surveyor's transit are $\frac{1}{3}$ of a

degree. A vernier is to be made for use with this circle and to read to $10''$. What is the length of the vernier and what is the number of divisions upon it?

3. Reduce an angle of $20°$ to radians. Reduce an angle of one radian to degrees.

4. The density of alcohol is 6.35 pounds per gallon. What is its density in pounds per cubic inch? What is its density in grams per cubic centimeter?

Note. — One gallon equals 231 cubic inches. One inch equals 2.54 centimeters. One pound equals 453.6 grams.

5. The density of iron is 7.8 grams per cubic centimeter. What is its density in pounds per cubic inch? What is the mass in pounds of an iron bar 30 feet long and $5\frac{1}{2}$ square inches sectional area?

6. A bottle weighs 50.62 grams empty, and 288.93 grams full of water at $21°$ C. What is the cubic contents of the bottle? The bottle full of oil at $21°$ C. weighs 239.2 grams. What is the specific gravity of the oil? What is the density of the oil? Neglect the effect of buoyant force of the air.

Note. — See chapter on thermometry for table of densities of water at various temperatures.

7. A clock with a pendulum which is supposed to beat seconds reads $12^h, 0^m, 16^s.8$ at mean noon on one day, and it reads $12^h, 1^m, 2^s.3$ at mean noon ten days later. What number of beats does the pendulum of the clock make in a mean solar day? What is the true value in hours, minutes, and seconds of an interval of time at the beginning of which the above clock reads, $5^h, 6^m, 10^s.2$, and at the end of which the clock reads $8^h, 33^m, 56^s.7$?

CHAPTER III.

PHYSICAL ARITHMETIC.

11. Measures; units. — In the expression of a physical quantity two factors always occur, a numerical factor and a unit. The numerical factor is called the *measure* of the quantity. Thus a certain length is 65 centimeters, a certain time interval is 250 seconds, a certain electric current is 25 amperes, a certain electromotive force is 110 volts.

It is a great help towards a clear understanding of physical calculations to consider that both *units* and *measures* are involved in a product of two physical quantities or in a quotient of two physical quantities. Thus a rectangle is 5 centimeters wide and 10 centimeters long; and its area is 5 centimeters times 10 centimeters, which is equal to 50 *square centimeters*. A cylinder is 10 centimeters long and the area of one of its ends is 25 square centimeters; and its volume is 10 centimeters times 25 square centimeters, which is equal to 250 *cubic centimeters*. A train travels 500 feet in 10 seconds and its average velocity during the time is 500 feet divided by 10 seconds which is equal to 50 *feet per second*. A body is dragged through a distance of 15 feet by a force of 10 pounds and the amount of work done is 15 feet times 10 pounds, which is equal to 150 *foot-pounds*. The word *per* connecting the names of two units indicates that the unit following is a *divisor*, thus a velocity of 50 feet *per* second may be and often is written 50 feet/second. A hyphen connecting the names of two units indicates a *product* of the units; products and quotients of units arrived at in this way are always *new* physical units. Thus the foot per second is a unit of velocity, the foot-pound is a unit of work.

It is important to carry the units through with every numerical calculation, the arithmetical operations among the various units

being indicated algebraically. When this is done there can be no ambiguity as to the meaning of the result, and when this is not done the result has, strictly speaking, no physical meaning at all.

Although the unit in terms of which a result is expressed is known * when the units are carried through a numerical calculation, it frequently happens that the unit is so entirely novel that it might almost as well be unknown. Thus the rule for finding the area of a rectangle by taking the product of length and breadth is entirely general, no matter what units of length are used, and the area of a rectangle 2 meters long and 50 centimeters wide is equal to 2 meters × 50 centimeters or 100 meter-centimeters. Now the *meter-centimeter* is a unit of area equal to the area of a rectangle one meter long and one centimeter wide and it is so entirely unfamiliar as a unit of area among men engaged in practical work that one might almost as well not know the value of an area at all as to have it given in terms of such a unit. **It is, for this reason, nearly always necessary to reduce the data of a problem to certain accepted units before these data can be used intelligibly in numerical calculations.**

12. Units, fundamental and derived. — The *fundamental physical units* are those which are fixed by arbitrary preserved standards. Thus the unit of length is preserved as a platinum bar in Paris, the unit of mass is preserved by a piece of platinum in Paris, and the second is naturally preserved in the constancy of speed of rotation of the earth.

Derived physical units are those which are defined in terms of the fundamental units and of which no material standard need be preserved. Thus the unit of area is defined as the area of a square of which each side is a unit of length, and there is no need of preserving a material standard of the unit of area. The unit of velocity is defined as unit distance traveled per second, and there is no need of preserving a material standard of the unit of velocity, indeed, it would be impracticable to preserve a velocity.

* See Art. 14, on dimensions of derived units.

Remark 1. — Quantities such as area, volume, velocity, electric current, etc., for which derived units are used, may be called *derived quantities* for the reason that they are defined (as quantities) in terms of the *fundamental quantities*, length, mass, and time. For example, the density of a body is defined as the ratio of its mass to its volume; the velocity of a body is defined as the quotient obtained by dividing the distance traveled during an interval of time, by the interval, etc.

Remark 2. — There is much latitude in the choice of fundamental units. A single fundamental unit would be theoretically sufficient, inasmuch as it is possible to define all physical units in terms of any one. The choice of fundamental units is a matter which is governed solely by practical considerations; in the first place the fundamental units must be easily preserved as material standards, and in the second place the fundamental quantities must be susceptible of very accurate measurement, for the definition of a derived unit cannot be *realized* * with greater accuracy than the fundamental quantities can be measured.

13. The c. g. s. system of units.† — Derived units based upon the *centimeter* as the unit length, the *gram* as the unit mass, and the *second* as the unit time, are in common use. This system of derived units is called the c. g. s. (centimeter-gram-second) system.

Thus the square centimeter is the c. g. s. unit of area, the cubic centimeter is the c. g. s. unit of volume, one gram per cubic centimeter is the c. g. s. unit of density, one centimeter per second is the c. g. s. unit of velocity, etc.

* The definition of a physical quantity is always an actual physical operation. Thus the mass of a body is defined by the operation of weighing with a balance; the density of a body is defined by the operations involved in the finding of mass and volume, for mass and volume must be determined before mass can be divided by volume to give density.

† So long as the English units of length and mass continue to be used, it will be necessary for engineers to use the units of the f. p. s. (foot-pound-second) system to some extent, although these systematic f. p. s. units are, many of them, never used in commercial work. Thus the f. p. s. unit of force is the *poundal*, the f. p. s. unit of work is the *foot-poundal*.

Practical units. — In many cases the c. g. s. unit of a quantity is either inconveniently small or inconveniently large so that the use of the c. g. s. unit would involve the use of very awkward numbers. Thus the power required to drive a small ventilating fan is 500,000,000 ergs per second, the electrical resistance of an ordinary incandescent lamp is 220,000,000,000 * c. g. s. units of resistance, the capacity of an ordinary Leyden jar is 0.000,000,000,000,000,005 c. g. s. units of capacity. In such cases it is convenient to use a multiple or a sub-multiple of the c. g. s. unit as a *practical* unit. Thus 5×10^8 ergs per second is equal to 50 watts, 22×10^{10} c. g. s. units of resistance is 220 ohms, 5×10^{-18} c. g. s. units of electrostatic capacity is equal to 0.005 micro-farad.

Legal units. — The system of units now in general use presents several cases in which the fundamental measurement of a derived quantity in terms of length, mass, and time is extremely laborious and not very accurate at best. Thus the measurement of electrical resistance in terms of length, mass, and time is very difficult, whereas the measurement of electrical resistance in terms of the resistance of a given piece of wire is very easy indeed and it may be carried out with great accuracy. In every such case the fundamental measurement is carried out once for all with great care and the best possible material copy is made of the derived unit and this copy is adopted as the standard legal unit.

14. Dimensions of derived units. — The definition of a derived unit always implies an equation which involves the derived unit together with one or more of the fundamental units of length, mass, and time. This equation solved for the derived unit is said to express the dimensions of that unit.† Thus the velocity of a body is defined as the quotient l/t, where l is the distance traveled by the body during the interval of time t, so that the unit of velocity is equal to the unit of length divided by the unit of time.

Examples. — Let l be the unit of length, m the unit of mass, and t the unit of time. Then the unit of area is equal to l^2, the unit of volume is equal to l^3, the unit of density is equal to m/l^3

* In the writing of very large or very small numbers it is always more convenient and more intelligible to use a positive or negative power of 10 as a factor. Thus 220,000,000,000 is best written as 22×10^{10}, and 0.000,000,000,000,000,005 is best written as 5×10^{-18}.

† And also the dimensions of the derived quantity.

the unit of velocity is equal to l/t, the unit of force is equal to ml/t^2, the unit magnetic pole is equal to $\sqrt{ml^3}/t$, etc.

Naming of derived units.—Many derived units have received specific names. Such are the *dyne*, the *erg*, the *ohm*, the *ampere*, the *volt*, etc. Those derived units which have not received specific names are specified by writing, or speaking-out, their dimensions. Thus the unit of area is the *square centimeter*, the unit of density is the *gram per cubic centimeter*, the unit of velocity is the *centimeter per second*, the unit of momentum is the *gram-centimeter per second* (written gr. cm./sec.). In the case of units which have complicated dimensions this method is not convenient in speech. Thus we specify a certain magnetic pole as 150 gr.$^{\frac{1}{2}}$ cm.$^{\frac{3}{2}}$/sec. (spoken, 150 *c. g. s. units pole*).

15. Scalar and vector quantities.—A *scalar quantity* is a quantity which has magnitude only. Thus everyone recognizes at once that to specify 10 cubic meters of sand, 25 kilograms of sugar, 5 hours of time, is, in each case, to make a complete specification. Volume, mass, time, energy, electric charge, etc., are scalar quantities.

A *vector quantity* is a quantity which has both magnitude and direction, and to specify a vector one must give both its magnitude and direction. This necessity of specifying both the magnitude and direction of a vector is especially evident when one is concerned with the relationship of two or more vectors. Thus if one travels a stretch of 10 kilometers and then a stretch of 5 kilometers more, he is by no means necessarily 15 kilometers from home; his position is, in fact, indeterminate until the direction of each stretch is specified. If one man pulls on a car with a force A of 200 units and another pulls with a force B of 100 units, the total force acting on the car is by no means necessarily equal to 300 units. In fact, the total force is unknown both in magnitude and direction until the direction as well as the magnitude of each force A and B is specified. Length, velocity, acceleration, momentum, force, magnetic field intensity, etc., are vector quantities.

Representation of a vector by a line.—In all discussions of physical phenomena which involve the relationships of vectors, it is a great help to the understanding to represent the vectors by lines. Thus in the discussion of the combined action of several forces on a body, it is a great advantage to represent each force by a line. To represent a vector by a line, draw the line in the direction of the vector (from any convenient point) and make the length of the line proportional to the magnitude of the vector. Thus if a northward velocity of 600 centimeters per second of a moving body is to be represented by a line, draw the line to the north and let each unit length of line represent a chosen number of units of velocity.

When a vector α is represented by a line, the line is parallel to α and the value of α is given by the equation

$$\alpha = S \cdot l$$

in which l is the length of the line and S is the number of units of α represented by each unit length of the line. The quantity S is called the *scale* to which the line represents the vector α.

16. Addition of vectors. The addition polygon. — Many cases arise in physics where it is necessary to consider the *single force* which is equivalent to the combined action of several given forces; where it is necessary to consider the *single actual velocity* which is equivalent to several given velocities each produced, it may be, by a separate cause; where it is necessary to consider the *single actual intensity* of a magnetic field due to the combined action of several causes each of which would alone produce a magnetic field of given direction and intensity; and so on. *The single vector is in each case called the vector-sum, or resultant, of the several given vectors.* Scalar quantities are added by the ordinary methods of arithmetic, thus 10 pounds of sugar plus 15 pounds of sugar is 25 pounds of sugar; but the addition of several vectors is not an arithmetical operation, it is a geometrical operation, and it is for this reason that the addition of vectors is sometimes called geometric addition. In order to make the

following discussion easily intelligible it is made to refer to the special case of the addition of forces.

Addition of two forces. The parallelogram of forces. — Let the lines *a* and *b*, Fig. 9, represent two forces acting upon any body,

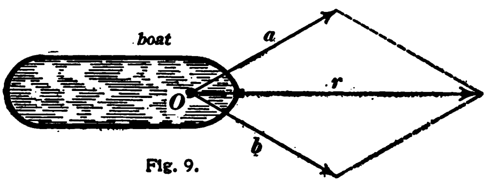


Fig. 9.

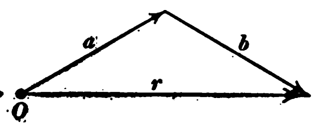


Fig. 10.

a boat for example. The vector-sum or resultant of the two forces *a* and *b* is represented by the diagonal *r* of the parallelogram of which *a* and *b* are the sides. It is evident that the geometric relation between *a*, *b* and *r* is completely represented by the triangle in Fig. 10, in which the line which represents the force *b* is drawn from the extremity of the line which represents the force *a*.

*Addition of any number of forces. The force polygon.** — Given a number of forces *a*, *b*, *c* and *d*. Draw the line which represents the force *a* from a chosen point *O*, Fig. 11, draw the line which represents the force *b* from the extremity of *a*, draw the line which represents *c* from the extremity of *b*, and draw the line which represents the force *d* from the extremity of *c*. Then the line from *O* to the extremity of *d* represents the geometric sum of the forces *a*, *b*, *c* and *d*.

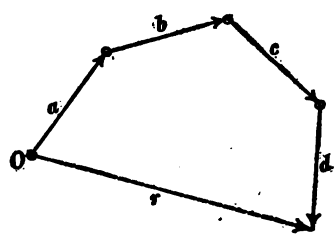


Fig. 11.

The vector sum of a number of forces is equal to zero if the forces are parallel and proportional to the sides of a closed polygon, the

* The sides of a force polygon need not all lie in a plane except of course when the polygon is a triangle. Three forces in equilibrium must not only be parallel to a certain plane, but their lines of action must actually lie in one plane, otherwise the forces will have an unbalanced torque action.

directions of the forces being in the directions in which the sides of the polygon would be traced in going round the polygon.

A particular case of this general proposition is that the vector sum or resultant of three forces is equal to zero if the three forces are parallel and proportional to the three sides of a triangle and in the direction in which the sides would be passed over in going round the triangle.

Note. — What is said above concerning the addition of forces applies to the addition of vectors of any kind, velocities, accelerations, magnetic field intensities and so on.

17. Resolution of vectors. — Any vector may be replaced by a number of vectors of which it is the sum. The simplest case is that in which a vector is replaced by two vectors which are parallel and proportional to the respective sides of a parallelogram, of which the diagonal represents the given vector. If a rectangle

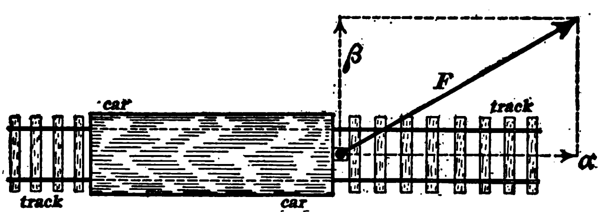


Fig. 12.

be constructed whose diagonal represents a given vector, then the sides of the rectangle will represent what are called the *rectangular components* of that vector in the directions of those sides.

Example. — Consider a force F, Fig. 12, pulling on a car which is constrained to move along a track. The force F is equivalent to α and β combined. The force β has no effect in moving the car, so that α is the effective part of F. This force α is called the *resolved part* of F, or the *component* of F in the direction of the track.

18. Scalar and vector products and quotients. — The product, or quotient, of a scalar and a vector, α, is another vector parallel to α.

Examples. — The distance l traveled by a body in time t is equal to the product vt, where v is the velocity of the body; and l is parallel to v.

The force F with which a fluid pushes on an exposed area a is equal to the product pa where p is the hydrostatic pressure of the fluid (a scalar). The vector direction of an area is the direction of its normal and this is parallel to F.

A force F which does an amount of work W in moving a body a distance d (which is parallel to F) is equal to W/d.

The acceleration of a body is equal to $\Delta v/\Delta t$ where Δv is the increment of velocity in the time interval Δt; the acceleration is a vector which is parallel to Δv.

19. Vector products. — CASE I. *Parallel vectors.* — The product or quotient of parallel vectors is a scalar. Thus $W = Fd$, in which W is the work done by a force F acting through a distance d in its direction; $p = F/a$, in which p is the pressure in a liquid which exerts a force F on an exposed area a; $V = la$, in which V is the volume of a prism of base a and altitude l.

CASE II. *Orthogonal vectors.* — The product, or quotient, of two mutually perpendicular vectors is a third vector at right angles to both factors. Thus $a = lb$ and $b = a/l$, in which a is the area of a rectangle of length l and breadth b; $T = Fl$, in which T is the *moment* or *torque* of a force F, and l is its arm. The product of a vector and a line perpendicular thereto is called a *moment* of the vector.

CASE III. *Oblique vectors.* — The product of two oblique vectors consists of two parts, one of which is a scalar and the other is a vector. Consider two vectors, α

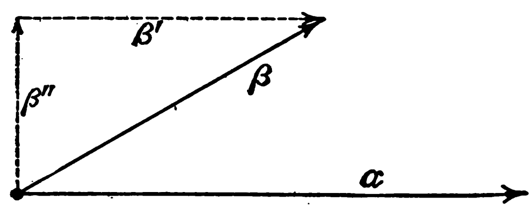


Fig. 13.

and β, Fig. 13. Resolve β into two components, β' and β'' respectively parallel to and perpendicular to α. Then

$$\alpha\beta = \alpha(\beta' + \beta'') = \alpha\beta' + \alpha\beta'',$$

in which $\alpha\beta'$ is a scalar and $\alpha\beta''$ is a vector. The scalar part of a vector product is indicated thus, $S \cdot \alpha\beta$ (read *scalar-alpha-beta*). The vector part of a vector product is indicated thus, $V \cdot \alpha\beta$ (read *vector-alpha-beta*). When $V \cdot \alpha\beta = 0$, α and β are parallel; when $S \cdot \alpha\beta = 0$, α and β are orthogonal.

Examples of products of vectors. — The area of any parallelogram is equal to $V \cdot bl$ where b and l are the two sides of the parallelogram. The work done by a force is equal to $S \cdot Fd$ where F is the force and d is the displacement of the point of application of the force. The volume of any parallopiped is equal to $S \cdot al$ where a is the area of the base and l is the length of the other edge. The torque action of any force is equal to $V \cdot Fr$ where F is the force and r is the lever arm.

20. Constant and variable quantities. The study of those physical phenomena which are associated with unvarying con-

ditions is comparatively simple, whereas the study of those phenomena which are associated with rapidly varying conditions is generally very complicated. Thus, to design a bridge is to so proportion its members that a minimum amount of material may be required to build it, and it is a comparatively simple problem to design a bridge to carry a steady load because it is easy to calculate the stress in each member due to a steady load, but it is an extremely complicated problem to design a bridge to carry a varying load, such as a moving locomotive which comes upon the bridge suddenly and moves rapidly from point to point.

Two kinds of variations are to be distinguished, namely, variations in space and variations in time.

Variations in time. — In the study of phenomena which depend upon conditions which vary in time, that is, upon conditions which vary from instant to instant, it is necessary to direct the attention to what is taking place at this or that instant, or, in other words, to direct the attention to what takes place during very short intervals of time, or, borrowing a phrase from the photographer, to make snap-shots, as it were, of the varying conditions.

*Definition of rate of change.** — *Principle of continuity.* In order to establish the rather difficult idea of instantaneous rate of change of a varying quantity, it is a great help to make use of a simple physical example. Therefore let us consider a pail out of which water is flowing through a hole in the bottom. Let

* Nearly everyone falls into the idea that such an expression as 10 feet per second means 10 feet of actual movement in an actual second of time, but a body moving at a velocity of 10 feet per second, might not continue to move for a whole second, or its velocity might change before a whole second has elapsed. Thus, a velocity of 10 feet per second is the same thing as a velocity of 864,000 feet per day, and a body need not move steadily for a whole day in order to move at a velocity of 864,000 feet per day. Neither does a man need to work for a whole month to earn money at the *rate* of 60 dollars per month, nor for a whole day to earn money at the *rate* of 2 dollars per day. A falling body has a velocity of 19,130,000 miles per century after it has been falling for one second, but to specify its velocity in miles per century does not mean that it moves as far as a mile or that it continues to move for a century! The units of length and time which appear in the specification of a velocity are completely swallowed up, as it were, in the idea of velocity, and the same thing is true of the specification of any rate.

x be the amount of water in the pail. Evidently x is a changing quantity. Let Δx be the amount of water which flows out of the pail during a given interval of time Δt, then the quotient $\Delta x/\Delta t$ is called *the average rate of change of* x during the given interval of time, and, if the interval Δt is very short, the quotient $\Delta x/\Delta t$ approximates to what is called *the rate of change of x at a given instant, or the instantaneous rate of change of x.* If the amount of water in the pail were to change by sudden jumps, as it were, then the rate of change of x at a given instant would be unthinkable; but *physical quantities which vary in value from instant to instant always vary continuously, and have, therefore, at each instant, a definite rate of change;* that is to say, the quotient $\Delta x/\Delta t$ always approaches a definite limiting value as the interval Δt is made shorter and shorter. The amount of water which flows out of a pail during a short interval of time is nearly proportional to the time; and if the time interval is very short, the amount of flow is more and more nearly in exact proportion to the time, so that the quotient $\Delta x/\Delta t$ approaches a perfectly definite finite value as Δt and Δx both approach zero. Let it be understood that this paying attention to what takes place during very short intervals of time does not refer to *observation* but to *thinking*, it is a matter of mathematics,* and therefore

* Two distinct methods are involved in the directing of the attention to what takes place during infinitesimal time intervals or infinitesimal regions of space.

(*a*) *The method of differential calculus.* — A phenomenon may be prescribed as a pure assumption and the successive instantaneous aspects derived from this prescription. Thus we may prescribe uniform motion of a particle in a circular path, and then proceed to analyze this prescribed motion as exemplified in Art. 33; or we may prescribe a uniform twist in a cylindrical metal rod, and then proceed to analyze the prescribed distortion. This method is also illustrated by the example given in the text, in which the expression for the growing sides of a square is prescribed.

(*b*) *The method of integral calculus.* — It frequently happens that we know the action that takes place at a given instant or in a small region, and can formulate this action without difficulty, and then the problem is to build up an idea of the result of this action throughout a finite interval of time, or throughout a finite region of space. For example, a falling body gains velocity at a known constant rate at each instant, how much velocity does it gain and how far does it travel in a given finite interval of time?

the following purely mathematical illustration is a legitimate example.

Example. Consider a square of which the sides are growing in proportion to elapsed time so that the length of each side of the square may be expressed as kt, where k is a constant and t is elapsed time reckoned from the instant when one of the sides is equal to zero. Then the area of the square is $S = k^2t^2$, and it is evident that the area is increasing. Let $t + \Delta t$ be written for t in the expression for S and we have

$$S + \Delta S = k^2(t + \Delta t)^2 = k^2t^2 + 2k^2t \cdot \Delta t + k^2(\Delta t)^2$$

whence, subtracting $S = k^2t^2$, member from member, we have

$$\Delta S = 2k^2t \cdot \Delta t + k^2(\Delta t)^2$$

or

$$\frac{\Delta S}{\Delta t} = 2k^2t + k^2 \cdot \Delta t$$

from which it is evident that $\Delta S/\Delta t$ becomes more and more nearly equal to $2k^2t$ as Δt is made smaller and smaller. The value of $\Delta S/\Delta t$ for an indefinitely small value of Δt is usually represented by the symbol * dS/dt so that we have

$$\frac{dS}{dt} = 2k^2t$$

Propositions concerning rates of change.—(a) Consider a quantity x which changes at a constant rate a, then the total change of x during time t is equal to at. For example, a man earns money at the rate of 2 dollars per day and in 10 days he earns 2 dollars per day multiplied by 10 days which is equal to 20 dollars. A falling body gains 32 feet per second of velocity every second, that is, at the constant rate of 32 feet per second per second, and in 3 seconds it gains 32 feet per second per second multiplied by 3 seconds which is equal to 96 feet per second.

* The symbol dS/dt is one single algebraic symbol and it is not to be treated otherwise. It stands for the *rate of change of S at any given instant* and is to be so read.

(*b*) Consider a quantity y which always changes k times as fast as another quantity x, then, if the two quantities x and y start from zero together, y will be always k times as large as x. That is, if $dy/dt = k \cdot dx/dt$, then $y = kx$ if y and x start from zero together.

Conversely, if one quantity is always k times as large as another it must always change k times as fast.

(*c*) Consider a quantity s which is equal to the sum of a number of varying quantities x, y and z, then the rate of change of s is equal to the sum of the rates of change of x, and y, and z. This may be shown as follows: let Δx, Δy and Δz be the increments of x, y and z during a given interval of time Δt, then the increment of s is

$$\Delta s = \Delta x + \Delta y + \Delta z$$

whence, dividing both members by Δt we have

$$\frac{\Delta s}{\Delta t} = \frac{\Delta x}{\Delta t} + \frac{\Delta y}{\Delta t} + \frac{\Delta z}{\Delta t}$$

or, if the interval Δt is very short, we have

$$\frac{ds}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} \qquad \text{(i)}$$

Conversely, if the relation (i) is given, that is, if s is a quantity whose rate of change is known to be equal to the sum of the rates of change of x, and y, and z, then s must be equal to $x + y + z$ if s, x, y, and z all start from zero together.

Variations in space. Imagine a bar of iron one end of which is red hot and the other end of which is cold. Evidently the temperature of the bar varies from point to point. The pressure in a vessel of water increases more and more with the depth, the density of the air decreases more and more with increasing altitude above the sea.

Such quantities as temperature, pressure, and density which refer to the physical conditions at various points in a substance

are called *distributed quantities*. The distribution is said to be *uniform* when the quantity has the same value throughout a substance, the distribution is said to be *non-uniform* when the quantity varies in value from point to point. Thus the temperature of the air in a room is uniform if it is the same throughout, whereas the temperature of a bar of iron which is red hot at one end and cold at the other end is non-uniform.

In the study of phenomena dependent upon conditions which vary from point to point in space, *the attention must be directed to what takes place in very small regions*, because too much takes place in a finite region. Let it be understood, however, that this paying attention to what takes place in small regions does not refer to *observation* but to *thinking*, it is a matter of mathematics, and therefore the following illustration is a legitimate example, even though the physics may not be entirely clear:

A rigid wheel rotates at a speed of n revolutions per second. Let us consider what is called the kinetic energy of the wheel. Now the kinetic energy of a moving body is equal to $\frac{1}{2}mv^2$, where m is the mass of the body in grams and v is its velocity in centimeters per second; but the difficulty here is that the different parts of the wheel have different velocities, and if we are to apply the fundamental formula for kinetic energy $(= \frac{1}{2}mv^2)$ to a rotating wheel it is necessary to consider each small portion of the wheel by itself. Thus, a small portion of the wheel at a distance r from the axis has a velocity which is equal to $2\pi nr$, and, if we represent the mass of the small portion by Δm, the kinetic energy of the portion will be $\frac{1}{2} \times \Delta m \times (2\pi nr)^2$, or $2\pi^2 n^2 \cdot r^2 \Delta m$; so that the total kinetic energy of the wheel will be equal to the sum of a large number of such terms as this. But, the factor $2\pi^2 n^2$ is common to all the terms, and therefore the total kinetic energy is equal to $2\pi^2 n^2$ times *the sum of a large number of terms like* $r^2\Delta m$. This sum is called the moment of inertia of the wheel.

Gradient. — Consider an iron bar of which the temperature is not uniform. Let ΔT be the difference of the temperatures at two points distant Δx from each other, then the quotient $\Delta T/\Delta x$

is called the *average* temperature grade or gradient along the stretch Δx, and if Δx is very small the quotient $\Delta T/\Delta x$ is called the actual temperature gradient and it is represented by the symbol dT/dx. The use of this idea of temperature gradient is illustrated in the discussion of the conduction of heat.

21. Varying vectors.* — The foregoing article refers solely to varying scalar quantities. The mathematics of varying vectors may also be considered in two parts, namely time variation and space variation.

Time variation of velocity.† — The velocity of a body is defined as the distance traveled in a given time divided by the time. When a velocity always takes place in a fixed direction it may be thought of as a purely scalar quantity. Thus a falling body has a velocity of 50 feet per second at a given instant and 3 seconds later it has a velocity of 146 feet per second, so that the increase of velocity in 3 seconds is 96 feet per second and the rate of increase is 96 feet per second divided by 3 seconds which is equal to 32 feet per second per second; but, suppose that the velocity of a body at a given instant is 50 units in a specified direction and that 3 seconds later it is 146 units in some other specified direction, then the change of velocity is by no means equal to 96 units and the rate of change of the velocity is by no means equal to 32 units per second.

The rate of change of the velocity of a body is called the *acceleration* of the body. Consider any moving body, a ball tossed through the air for example, let its velocity, v_1, at a given

*This branch of mathematics is completely ignored in present undergraduate courses, and yet no one can have a clear insight into the phenomena of motion without having an idea of the time variation of velocity, and no one can have a clear insight into the phenomena of fluid motion and of electricity and magnetism without having some understanding of the space variation of such vectors as fluid velocity, magnetic field, and electric field.

† What is here stated concerning the time variation of velocity applies to the time variation of any vector whatever. Thus if any varying vector is represented to scale by a line drawn from a fixed point, then the velocity of the end of the line represents the rate of change of the vector to the same scale that the line itself represents the vector.

instant be represented by the line OA, Fig. 14, let its velocity v_2 at a later instant be represented by the line OB, and let the elapsed time interval be Δt. Now the velocity which must be added (geometrically) to v_1 to give v_2 is the velocity Δv which is represented by the line AB. Therefore, the change of the velocity of the tossed ball during the interval Δt is the vertical velocity Δv shown in the figure, and the acceleration, a, of the ball is equal to $\Delta v/\Delta t$ which is of course in the direction of Δv. If the varying velocity of a tossed ball be represented by a line OP, Fig. 15, drawn from a fixed point O; then, as the velocity changes,

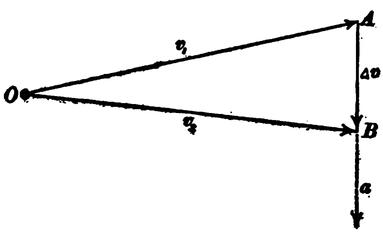


Fig. 14.

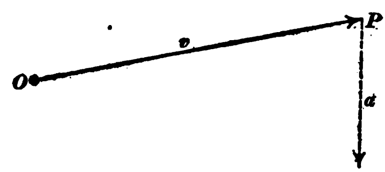


Fig. 15.

the line OP will change, the point P will move, *and the velocity of the point P will represent the* **acceleration of the body** *to the same scale that the line OP represents the* **velocity of the body.**

The orbit of a moving body is the path which the body describes in its motion. Thus the orbit of a tossed ball is a parabola as shown in Fig. 16*a*, and the orbit of a ball which is twirled on a cord is a circle as shown in Fig. 17*a*. Suppose a line OP, Fig. 16*b* or Fig. 17*b*, drawn from a fixed point, be imagined to change in such a way as to represent at each instant the velocity of the body as it describes its orbit, then the end P of the line will describe a curve called the *hodograph* of the orbit and, of course, the velocity of the point P will represent at each instant the acceleration of the moving body. Thus the hodograph of a tossed ball is a vertical straight line as shown in Fig. 16*b*, this is evident when we consider that a tossed ball has a constant *acceleration* vertically downwards, so that the point P, Fig. 16*b*,

must move at a constant *velocity* vertically downwards. The hodograph of a ball twirled on a string is a circle as shown in Fig. 17*b*, this is evident when we consider that the magnitude of

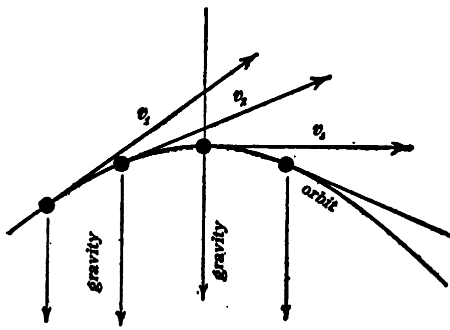


Fig. 16*a*.

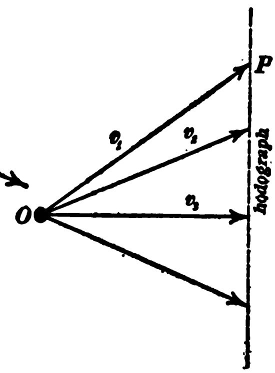


Fig. 16*b*.

the velocity of the body in Fig. 17*a* is constant, so that the length of *OP*, Fig. 17*b*, which represents the velocity of the body, must

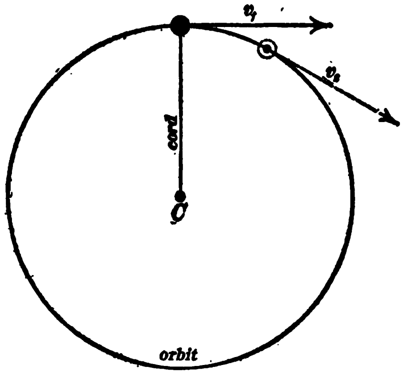


Fig. 17*a*.

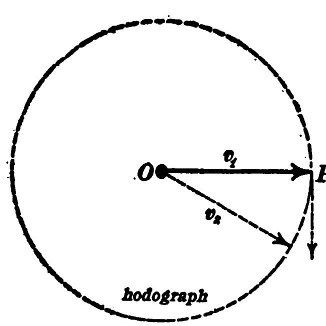


Fig. 17*b*.

also be constant. Furthermore, the velocity of the ball is always at right angles to the cord in Fig. 17*a*, and therefore the line *OP*, Fig. 17*b*, is always at right angles to the cord, so that the

generating point P of the hodograph makes the same number of revolutions per second as the twirled ball. It is important to note also that the velocity a of the point P in Fig. 17*b*, is always parallel to the cord in Fig. 17*a*, that is, the acceleration of the ball in Fig. 17*a* is at each instant in the direction of the cord.

Space variation of vectors. — The simplest idea connected with the space variation of a vector is the idea of the stream line, if it may be permitted to use the terminology of fluid motion to designate a general idea. A stream line in a moving fluid is a line drawn through the fluid so as to be at each point parallel to the direction in which the fluid is moving at that point. Thus if a pail of water be rotated about a vertical axis, the stream lines are a system of concentric circles as shown in Fig. 18, and Fig. 19 represents the approximate trend of the stream lines in the case of a jet of water

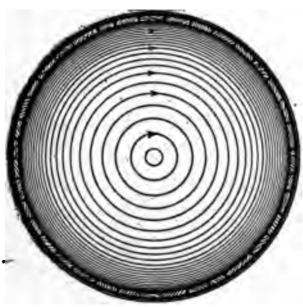


Fig. 18.

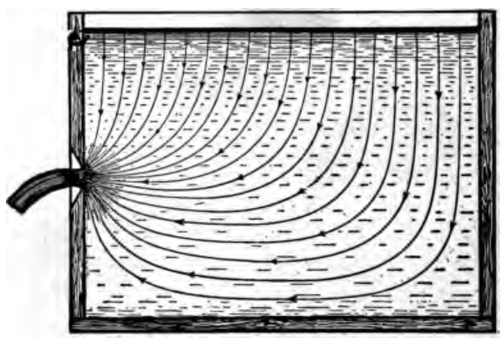


Fig. 19.

issuing from a tank. Where the stream lines come close together the velocity of the water is great and where they are far apart the velocity is small.

The potential of a distributed vector. — In some simple cases of fluid motion it is geometrically possible to look upon the stream lines as the lines of slope of an imagined hill the steepness of which represents the velocity of the fluid at each point both in magnitude and in direction. The height of this imagined hill at a point is called the *potential* of the fluid velocity at that point. The idea of potential is especially useful in the study of electricity and magnetism.

PROBLEMS.

8. A table top is 10 feet long and 50 inches wide. Find its area in inch-feet, and explain the result.

9. A body has a mass of 60 pounds and a volume of 2 gallons. Find its density without reducing data in any way.

10. A water storage basin has an area of 2,000 acres, find the volume of water in acre-feet required to fill the basin to a depth of 16 feet. Explain the acre-foot as a unit of volume and find the number of gallons in one acre-foot.

11. A man travels at a velocity of 6 feet per second; how far does he travel in two hours? Find the result without reducing the data in any way. Explain the *foot-hour per second* as a unit of length.

12. A man starts from a given point and walks three miles due north, then two miles northeast, then two miles south, and then one mile east. Show by means of a vector diagram how far, and in what direction he is from his starting point.

13. A stream flows due south at a velocity of two miles per hour. A man rows a boat in an eastward direction at a velocity of four miles per hour. What is the actual velocity of the boat and in what direction is it moving?

14. A stream flows due south at a velocity of two miles per hour. A man, who can row a boat at a velocity of four miles per hour, wishes to reach the opposite bank at a point due east of his starting point. Show by means of a vector diagram the direction in which he must row.

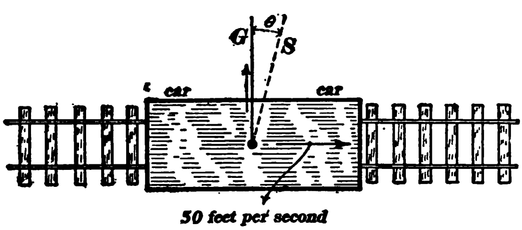


Fig. 16*p*.

15. An anemometer on board ship indicates a wind velocity of 28 miles per hour apparently from the northeast. The ship, however, is moving due north at a velocity of 15 miles per hour. What is actual direction and velocity of the wind?

16. A gun which produces a projectile-velocity of 200 feet per second is mounted aboard a car with its barrel G at right angles to the direction of motion of the car, as shown in Fig. 16*p*. The

car is traveling 50 feet per second. The sights are to be arranged at an angle θ to the gun barrel as shown. Find the value of θ so that the ball may hit any object which, at the instant of firing, is in the line S of the sights.

Note. — This problem illustrates the vector addition of velocities. The sight line must of course be in the actual direction in which the bullet is moving when it leaves the gun.

17. A body moves at a velocity of 20 miles per hour in a direction 20° north of east; find the northward and eastward components of its velocity.

18. Find the magnitude and direction of the single force which is equivalent to the combined action of three forces A, B and C; force A being northwards and equal to 200 units, force B being towards the north-east and equal to 150 units, and force C being eastwards and equal to 100 units.

19. A horse pulls on a canal boat with a force of 600 pounds-weight and the rope makes an angle of 25° with the line of the boat's keel. Find the component of the force parallel to the keel.

20. If 500 grams of water leak out of a pail in 26 seconds, what is the average rate of leak?

21. A man earns $27.50 in 8½ days. What is the average rate at which he earns money?

22. During 28 seconds the velocity of a train increases from zero to 12 feet per second. What is the average rate of increase of velocity?

23. A train gains a speed of 32 miles per hour in 80 seconds. Find its average acceleration in miles per hour per second.

24. A pole 22 feet long is dragged sidewise over a field at a velocity of 8 feet per second. At what rate does the pole sweep over area?

25. A prism has a base of 25 square cm., its height is increasing at the rate of 5 cm. per second. How fast is its volume increasing?

26. The slope of a hill falls 60 feet in a horizontal distance of 270 feet. What is the grade?

27. One side of a brick wall is at a temperature of $0°$ C. and and other side is at a temperature of $23°$ C. The wall is 30.5 cm. thick. What is the average temperature gradient through the wall?

28. At a given point in a water pipe the water pressure is 110 lbs. per square inch. Twenty-two feet from this point the pressure is 75 pounds per square inch. What is the average pressure gradient along the pipe?

CHAPTER IV.

SIMPLE STATICS.

22. Balanced force actions. — When a body remains at rest, or continues to move with uniform velocity along a straight path, or when a body continues to rotate at uniform speed about a fixed axis, the forces which act on the body are *balanced*. The study of balanced force actions, or as it is sometimes expressed, *the study of forces in equilibrium*, constitutes the science of *statics*. The science of statics really includes the study of equilibrium in its widest sense, namely, the equilibrium of the forces which act upon the parts of a distorted body (the statics of elasticity), and the equilibrium of the forces which act upon the parts of a fluid (hydrostatics); but these branches of statics are treated in subsequent chapters, and the present chapter deals only with the relationship between forces which do not tend to produce translatory motion or rotatory motion.*

Every one knows that even a single force acting on a body may cause both translation and rotation. Thus a boat which is pushed away from a landing generally turns more or less as it moves away; in this case, however, it may be the force action of the water on the boat that causes the turning, but a sidewise push on the bow of the boat certainly produces both translation and rotation irrespective of the force action of the water.

From the fact that even a single force can produce both translation and rotation, it may seem as though it would be impossible to consider separately the two effects of a force, namely, (*a*) tendency to produce translatory motion and (*b*) tendency to produce rotation; but every one knows that forces may produce translation without producing rotation, and that forces may produce rotation without producing translation. Thus a table may be moved without turning, and a top may be set spinning without

* See Art. 28.

any perceptible sidewise motion. The fact is, the two tendencies a and b must be considered separately.

23. Tendency to produce translatory motion. First condition of equilibrium. — In order that a number of forces may have no tendency to produce translatory motion, it is necessary and sufficient that the vector sum of the forces be equal to zero, that is, the forces must be parallel and proportional to the sides of a closed polygon and in the directions in which the sides of the polygon would be passed over in going round the polygon, as explained in Art. 16. This statement of the first condition of equilibrium leads directly to the graphical method of solving a problem in statics. When the algebraic method of solution is to be used, the following statement of the first condition of equilibrium is preferable.

Let each force be resolved into rectangular components parallel to chosen axes of reference. Let X_1, X_2, X_3, etc., be the x-components of the various forces, and let Y_1, Y_2, Y_3, etc., be their y-components. Then if the tendency of the forces to produce translatory motion is zero, we have

$$\left.\begin{aligned} X_1 + X_2 + X_3 + \text{etc.} &= 0 \\ \text{and} \qquad Y_1 + Y_2 + Y_3 + \text{etc.} &= 0 \end{aligned}\right\} \qquad (1)$$

It is necessary in forming these equations to consider x-components as positive when they are towards the right and negative when they are towards the left; and to consider y-components as positive when they are upwards and as negative when they are downwards.

24. Tendency to produce rotation. Definition of torque. — Consider a lever AB, Fig. 20, supported at the point O and in equilibrium under the action of two forces F and F' acting as shown, then Fa is equal to $F'a'$. The product Fa measures the tendency of the force F to turn the lever in one direction about O, and the product $F'a'$ measures the tendency of the force F' to turn the lever in the other direction about the point

O, and, rightly considered, one of these products should be considered as positive and the other as negative so that we may write

$$Fa + F'a' = 0$$

The product of a force and the perpendicular distance from a given point to the line of action of the force, measures the tendency of the force to turn a body about the given point, and this product is called the **moment of the force** *about the point, or its* **torque.**

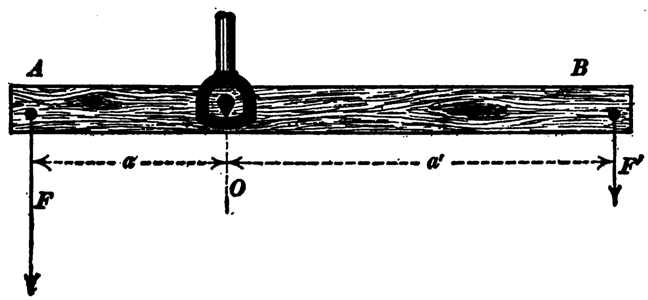


Fig. 20.

Second condition of equilibrium. — In order that a number of forces may have no tendency to turn a body it is *necessary* * that the sum of the torque actions of the forces about any chosen point be equal to zero, torques tending to turn the body in one direction being considered as positive and torques tending to turn the body in the opposite direction being considered as negative. The chosen point is called the *origin of moments.*

It is often convenient to express this second condition of equilibrium in terms of the components of the respective forces and the coördinates of their points of application. Consider a force F which acts on a body, Fig. 21, at the point p of which the coördinates are x and y as shown. Let X and Y be the components of F, then Xy is the torque action of the component X about the origin O, and Yx is the torque action of the component Y about O; and inasmuch as these torque actions are in opposite directions they must be considered as opposite in sign so

* This condition is also a sufficient condition if the first condition of equilibrium is satisfied.

that the total torque action of the force F about O is equal to $Xy - Yx$. The torque action of each force acting on a body

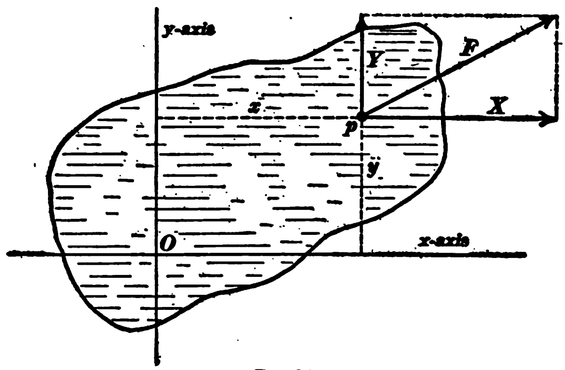


Fig. 21.

may be calculated in this way and then the sum of these torque actions must be equal to zero. That is

$$\Sigma(Xy - Yx) = 0 \qquad (2)$$

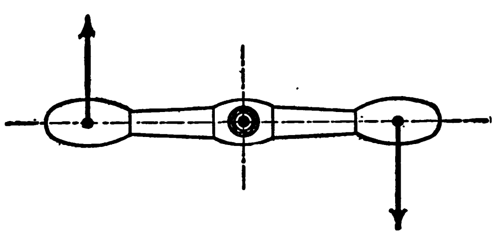


Fig. 22.

Pure torque. — A number of forces which act on a body may have no tendency to produce translatory motion and still have a tendency to produce rotation, or, in other words, a number of forces may satisfy the first condition of equilibrium and not satisfy the second condition. Such a combination of forces constitutes a pure torque and the total torque action is the same about any point whatever. For example, the two equal and opposite forces which are exerted on the handle of an auger, as shown in Fig. 22, constitute a pure torque. Such a pair of forces is sometimes called a *couple.*

25. Three forces in equilibrium intersect at a point. — The forces must lie in one plane in order to satisfy the first condition of equilibrium. Choose the origin of moments at the intersection of two of the forces. Then the third force must pass through this point, otherwise it will have an unbalanced torque action about this point.

Any number of forces, not in equilibrium, acting on a body are together equivalent to a single force, which is called their resultant; except when the forces constitute a pure torque.

Proof. — Given a number of forces in equilibrium. If *one* of these forces is omitted, the combined action of the others must be equivalent to an equal and opposite force having the same line of action. The exception is also evident, since by omitting one of a set of forces in equilibrium the others cannot constitute a pure torque. The magnitude and direction of the resultant of a number of forces is determined as their vector sum. The point of application of this resultant is the point about which the given forces have no torque action.

The center of mass of a body is the point of application of the resultant of the parallel forces with which gravity acts upon the particles of the body.

Proof. — Let the origin of coordinates be chosen at the center of mass of the body

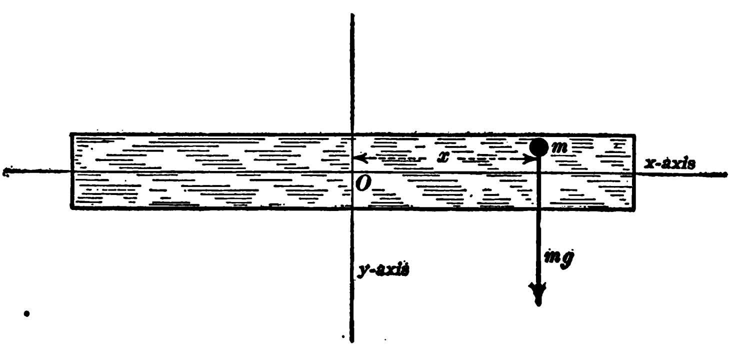


Fig. 23.

(see Art. 50) and let the y-axis be downwards as shown in Fig. 23. Consider a particle of mass m which is pulled downwards by the force mg (see Art. 33). The torque action of this force about the origin is mgx, and the torque action of all such forces is Σmgx or $g\Sigma mx$. But Σmx is equal to zero inasmuch as the origin is sup-

posed to be at the center of mass of the body. Therefore the torque action of all of the forces mg about O is zero, and consequently O is the point of application of the total force with which gravity acts on the body. The center of mass of a body is for this reason sometimes called the center of gravity of the body.

28. D'Alembert's principle. — The ease with which the relation between a number of forces in equilibrium can be shown, especially by the use of graphical methods, makes it desirable to extend the idea of balanced forces to the subject of dynamics. The principle on which this can be done was first enunciated by D'Alembert and it is called D'Alembert's principle. The following is a statement of D'Alembert's principle as applied to a particular case. A part of a mechanism moves in a prescribed way under the combined action of a number of given forces; if there be introduced into the system *fictitious forces* which are equal and opposite to the forces required to produce the known accelerations, then the system of given forces together with the fictitious forces will be in equilibrium. Thus if forces be imagined to act outwards on every part of a rotating metal hoop or ring, then these forces may be thought of as producing the tension in the rotating hoop, the hoop may be thought of as stationary, and the problem becomes a problem in statics. See Art. 60.

PROBLEMS.*

29. Three cylinders, each 12 inches in diameter and each weighing 200 pounds, lie in a rectangular trough of which the width is 27 inches. Assuming that the force action at each point of contact is at right angles to the surface of contact, find the force action at each point of contact of the cylinders with each other and with the trough.

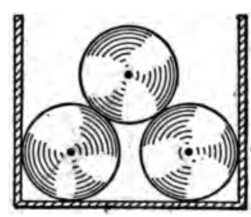


Fig. 29p.

30. The steam in an engine cylinder pushes on the piston with a force of 12,000 pounds-weight. The positions and lengths of connecting rod and crank are shown in Fig. 30p. Find the force with which the cross-head pushes sidewise against the guide, the thrust of the connecting rod, and the torque in pound-feet exerted on the crank-shaft, neglecting friction throughout.

* Several problems in statics, illustrating D'Alembert's principle and the principle of virtual work, are given at the end of Chapter VI. Unquestionably it is more instructive to solve problems in statics by the graphical method than by the analytical method, and no case exists in practice for which the graphical method is not sufficiently accurate. Most of the problems in this chapter should be solved by the graphical method, and, if desirable, the analytical solution may then be required as an additional exercise.

31. A rope 15 feet long supports a 1,000-pound weight from two supports SS, as shown in Fig. $31p$. Find the tension in

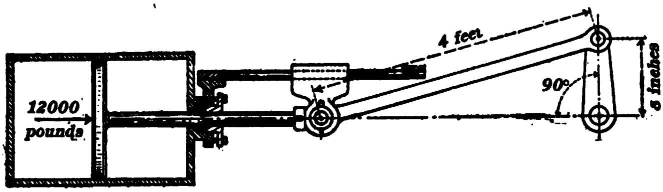


Fig. $30p$.

each part of the rope and the vertical force on each support, neglecting weight of rope.

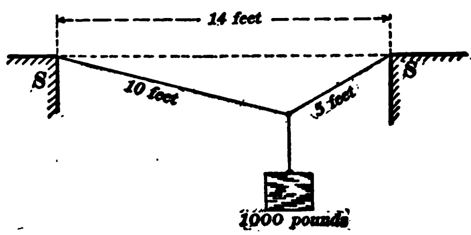


Fig. $31p$.

32. A right-angled lever of which the position and dimensions are shown in Fig. $32p$ carries a weight of 200 pounds at the end of the 24-inch arm. Find the value and direction of the force exerted on the lever by the point O, and find the value of the force F.

Note. — Three forces in equilibrium intersect in a point so that the line of action of the unknown force at O passes through the point where the lines W and F intersect.

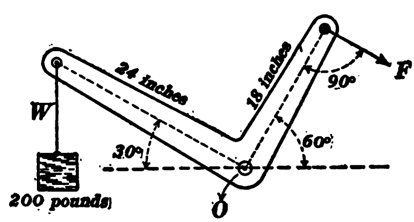


Fig. $32p$.

33. Find the force F required to draw a 4,000-pound wagon up an inclined plane of the dimensions shown in Fig. $33p$, neglecting friction.

34. Find the force F, Fig. 34p, required to draw a 200-pound block up an inclined plane of the dimensions shown, the coefficient of friction between block and plane being 0.2.

Note. — See Art. 52 for an explanation of the coefficient of friction. The direction of the force which the plane exerts on the moving block is PO, as shown in Fig. 34p.

35. A wedge of which the shape is indicated in Fig. 35p, is pushed between two blocks A and B with a force F of 5,000 pounds-weight. The coefficient of friction between the wedge

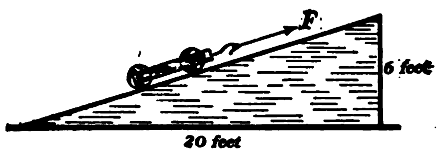


Fig. 33p.

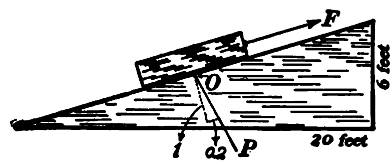


Fig. 34p.

and the blocks is 0.2. Find the components at right angles to F of the forces with which the wedge pushes on A and B.

36. A block weighing 100 pounds rests on a smooth floor, the

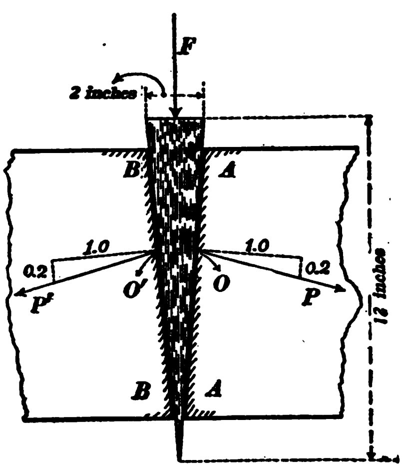


Fig. 35p.

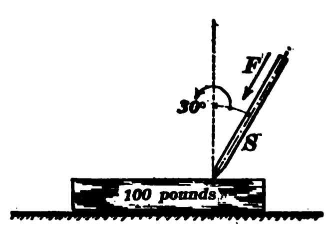


Fig. 36p.

coefficient of friction between the block and the floor is 0.2, and the block is pushed along by a stick S, as shown in Fig. 36p. Find the thrust of the stick in pounds-weight.

37. A ladder 16 feet long and weighing 100 pounds has its center of gravity 7 feet from its lower end which rests on a floor at a distance of 4 feet from a vertical wall against which the ladder rests, as shown in Fig. 37*p*. Assuming the force *a* with which the wall pushes on the ladder to be horizontal, find the

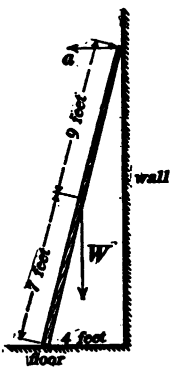


Fig. 37*p*.

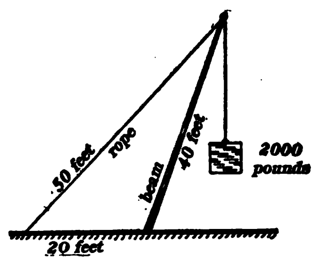


Fig. 38*p*.

magnitude of *a* and the direction and magnitude of the force with which the ladder pushes against the floor.

38. A forty-foot beam arranged as shown in Fig. 38*p* supports

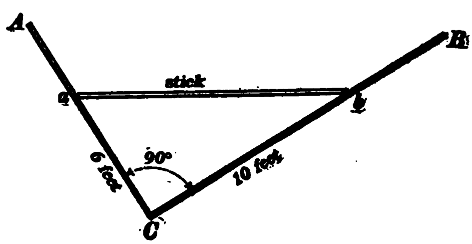


Fig. 39*p*.

a weight of 2,000 pounds. Find the pull of the rope and the thrust of the beam.

39. A stick *ab*, Fig. 39*p*, lies across a right-angled trough *ABC* as shown. Find the line in the figure which must be vertical in order that the stick may be in equilibrium on the assumption that there is no friction at *a* and *b*.

Note. — The force of gravity on the stick is a vertical force and its point of application is at the center of the stick. Three forces in equilibrium intersect in a point.

40. Suppose that the coefficient of friction at a and b, Fig. 39*p*, is 0.1, find the limiting positions of the vertical between which the stick will be in equilibrium.

41. A uniform stick 6 feet long which weighs 10 pounds has a 15-pound weight hung 1 foot from one end, a 20-pound weight hung 2 feet from the same end, and a 25-pound weight hung 6 inches from the other end. Find the point at which the stick can be supported in a horizontal position.

Note. — To solve this problem, choose one end of the stick as the origin of moments and let x be the distance from this end to the point of application of the resultant of all of the forces. Then the sum of the moments of the several forces about the origin is equal to the sum of the forces multiplied by x, from which relation x can be calculated.

42. A simple bridge truss consists of two struts and a tie rod as shown in Fig. 42*p*. A weight W of 2,000 pounds hangs from

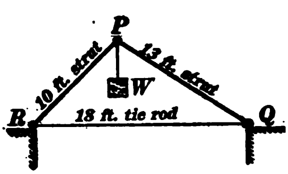


Fig. 42*p*.

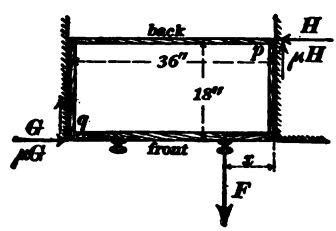


Fig. 43*p*.

the point P. Find the compression in each strut, the vertical pressure on each abutment, and the tension in the tie rod, neglecting weight of the parts of the truss.

Note. — To solve this problem, consider first the equilibrium of the three forces at P, two of which forces are parallel to the respective struts; then consider the equilibrium of the three forces at Q; and then consider the equilibrium of the three forces at R. Assume the abutments to exert vertical forces on the truss.

43. A table drawer, 36 inches in breadth and 18 inches in depth (front to back), is pulled by a force F applied at a distance x from one corner as shown in Fig. 43*p*. The drawer binds at the two corners p and q, and it is required to find the smallest

value of x for which the drawer can be pulled out by the force F; the coefficient of friction between drawer and guides being 0.2.

Note.—At p the guide exerts upon the drawer the force H and another force which cannot exceed μH, μ being the coefficient of friction between drawer and guides. At q the guide exerts upon the drawer, a force G and another force μG. Now when it is just possible to pull out the drawer, the various forces F, G, H and the full value of the forces μH and μG are in equilibrium. The algebraic conditions of equilibrium are:

Fig. 44*p*.

1. That the sum of all forces to the right be equal to the sum of all the forces to the left. That is

$$G = H \tag{i}$$

2. That the sum of all the downward forces be equal to the sum of all the upward forces. That is

$$F = \mu G + \mu H \tag{ii}$$

3. That the sum of all the right-handed torque actions about any chosen origin of moments be equal to the sum of all the left-handed torque actions. Choosing the point q as the origin of moments, this condition gives

$$\underset{\text{Right-handed torques.}}{F(b - x)} = \underset{\text{Left-handed torques.}}{b\mu H + aH} \tag{iii}$$

in which a is the depth of the drawer front to back, and b is its width.

All three unknown forces F, G and H may be eliminated from these equations and the value of x determined in terms of b, a and μ.

This problem should be solved graphically as well as algebraically. An interesting modification is to find the direction of F when x is given.

44. Given a tackle block arranged as shown in Fig. 44*p*. Find the weight W which can be lifted by a force F equal to 150 pounds-weight, neglecting friction.

Note.—The simplest argument of this problem is as follows: The tension of the rope is everywhere equal to 150 pounds-weight if friction is negligible. Therefore, the four strands of rope which lead to the lower block exert a total lifting force of 600-pounds weight.

CHAPTER V.

DYNAMICS. TRANSLATORY MOTION.

27. Force and its effects. — Our fundamental notions of force arise from the sensations which are associated with muscular effort, and the effects of force are extremely varied. Thus a force may break a body or distort it. A force exerted in rubbing one body against another produces heat. When ice is compressed it melts, when vapor is compressed it condenses to liquid. When an unbalanced force acts on a body the velocity of the body changes. The fact is that nearly every physical phenomenon involves force action of one kind or another.

A force can be measured only in terms of its effects. The effect which can be most easily used for the measurement of force is the effect the force has in distorting a body, as, for example, in stretching a helical spring. *The simplest effect of a force, however, is the change which the force, when unbalanced, produces in the velocity of a body, inasmuch as this effect is independent of the nature of the body.* This effect is now universally adopted as the effect by which a force is measured.

The study of the effects of unbalanced forces in modifying the motion of bodies constitutes the science of *dynamics.*

28. Types of motion. — Motion is infinite in variety, and there are certain simple types of motion, the discussion of which constitutes the science of mechanics. Thus we have *translatory motion* in which every line in the moving body remains unchanged in direction, *rotatory motion* in which a certain line in the moving body remains fixed, *oscillatory motion* in which the moving body undergoes periodic changes of shape, *wave motion* in which a localized pulse of motion travels through the body, and *simple motion of flow* of fluids. In general, a body may not only perform all of these types of motion simultaneously, but several varieties of each type may coexist. It is necessary, however, to

study each type of motion by itself, and some help is afforded towards the keeping of the several types of motion clearly separated in one's mind by conceiving of an ideal body which can perform only one or another type of motion, as follows :

A *material particle* is an ideal body so small that the only sensible motion of which it is capable is translatory motion. One must not, however, lose sight of the fact that the term, *material particle*, is used merely to rivet one's attention to translatory motion, and any body whatever which performs translatory motion, or which is acted upon by forces which tend to produce translatory motion only, may be thought of as a particle if one wishes to think in such terms.

A *rigid body* is an ideal body which cannot alter its shape and which is capable only of translatory motion and rotatory motion. One must not, however, lose sight of the fact that the term *rigid body* is used merely to exclude the idea of change of shape in the discussion of rotatory motion.

Bodies assumed to possess ideal elastic properties are useful in describing some of the more important phenomena of oscillatory motion and wave motion. For example, the *isotropic body* which has the same properties in all directions ; the *perfectly elastic body* which is assumed to follow Hooke's law exactly ; the *perfectly flexible string* which is useful in describing the oscillatory motion of strings, and so on.

An *incompressible frictionless fluid* is an ideal fluid which cannot be reduced in volume, and which, once in motion, would remain in motion indefinitely. The idea of an incompressible frictionless fluid is useful in describing some of the more important aspects of fluid motion.

The use of ideal bodies and substances in the development of mechanics may seem to be objectionable, but it is necessary to discuss one thing at a time, and it is even more necessary to ignore the interminable array of minute effects which always accompany every physical phenomenon, any detailed consideration of which would complicate every engineering problem beyond the possi-

bility of a practical solution. Thus one may describe in a general way the motion of a railway train along a straight level track by specifying its velocity, whereas the actual motion involves the swaying and vibration of the cars and the rattling of every loose part, it involves a complicated phenomenon of motion which is called journal friction, and it involves the yielding of the track and a whirling, eddying motion of the air, it is, in fact, infinitely complicated, and the railway engineer who, for example, is concerned merely with the design of a locomotive of adequate power, sums up all of these effects in a rough estimate of the total frictional drag which the locomotive has to overcome.

29. Types of force action. — To each type of motion there is a corresponding type of force action. Thus a force action which tends to produce translatory motion only, is called a *linear force*, and a force action which tends to produce rotatory motion is called a twisting force or *torque.* The internal force actions in a distorted body which is oscillating or which is transmitting wave motion, are resolvable into what are called *hydrostatic pressure*, *longitudinal stress*, and *shearing stress.* The force action which tends-to produce the obscure motion which constitutes the electric current is called *electromotive force.* This mere enumeration of the various types of force action is intended only to emphasize the classification of motion into the various types mentioned in the previous article.

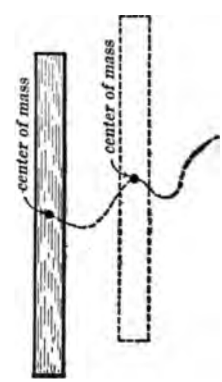


Fig. 24.

30. Translatory motion. Center of mass. — When a body moves so that every line in the body remains unchanged in direction the body is said to perform translatory motion. The simplest case of translatory motion is the motion of a body along a straight path, either with constant or varying velocity, as exemplified by the motion of a car along a straight track or the motion of a ship on a straight course. The most general case of translatory motion,

however, is where a given point of a body describes any path whatever in any way whatever, but where every line in the body remains unchanged in direction, as indicated in Fig. 24.

Grasp a long slim stick at its middle and move it up and down and to and fro in any way, but without changing the direction of the stick; it seems, with the eyes closed, as if the stick were a heavy body concentrated in the hand, or in other words the stick behaves as if it were all concentrated at its middle point. *That point in a body at which a single force must be applied to produce translatory motion* is called the *center of mass** of the body. The center of mass of a uniform stick is at the middle of the stick.

31. Displacement, velocity, and acceleration of a particle. — When a particle moves from one position to another it is said to be displaced and the distance (and direction) from the initial to the final position of the particle is called the *displacement.*

The displacement of a particle divided by the time during which the displacement takes place is called the *average velocity* of the particle. Inasmuch as the particle may move in any way whatever in making a given displacement, it is evident that the actual velocity of the particle at successive instants during the displacement may be very different from the average velocity; but if the interval of time be extremely short (and, of course, the displacement small) then all irregularities vanish, in accordance with the principle of continuity as stated in Art. 20, and therefore the average velocity during a very short interval of time is the *actual velocity* of the particle at the given instant, that is, during the very short interval.

When the velocity of a particle is changing, the actual change during a given interval of time divided by the interval is called the *average acceleration* of the particle during the interval, and if the interval is very short the acceleration so defined is the *actual acceleration* at the given instant.

The actual velocity of a particle at a given instant is, of course, never determined by any attempt to observe the displacement

* Sometimes called *center of gravity.*

during a very short interval of time, and the actual acceleration of a particle at a given instant is, of course, never determined by any attempt to observe the change of velocity during a very short interval of time. The only importance to be attached to the above definitions is that the student should see that they are legitimate, so that the ideas may be used intelligibly.*

32. Newton's laws of motion.

I. All bodies persevere in a state of rest, or in a state of uniform motion in a straight line, except insofar as they are made to change that state by the action of an unbalanced force.

II. (*a*) The acceleration of a particle is parallel and proportional to the unbalanced force acting on the particle.

(*b*) The acceleration which is produced by a given unbalanced force is inversely proportional to the mass of the particle.

III. Action is equal to reaction and in a contrary direction.

(1) The first law describes the behavior of a particle upon which no unbalanced force acts. The behavior is simply that *the velocity of the particle does not change*, and, conversely, if a body moves at uniform velocity in a straight line, the forces which act upon it are balanced.

(2′) The second law describes the behavior of a particle when acted upon by an unbalanced force. The behavior is simply that *the particle gains velocity in the direction of the force at a rate which is* (*a*) *proportional to the force and* (*b*) *inversely proportional to the mass of the particle.*

When Newton expounded this fact he evidently had it in mind that a force is measured by some effect other than acceleration, otherwise he could not have affirmed, except as a definition, that the acceleration which a force produces is proportional to the force. The production of acceleration is now adopted as the effect by which forces are measured.

(2″) The second law may be further paraphrased as follows: (*a*) The amount of velocity gained by a given particle in a given

* The discussion of the time variation of velocity which is given in Art. 21 should be reviewed at this point.

interval of time is proportional to the unbalanced force acting on the particle, and the gained velocity is parallel to the force. (*b*) The amount of velocity produced by a given unbalanced force in a given interval of time is inversely proportional to the mass of the particle upon which the force acts, and the gained velocity is, of course, parallel to the force.*

(3) The third law expresses the fact that a force is always due to the mutual action of two bodies, that this mutual action always consists of a pair of equal and opposite forces, and that one of these forces acts on body number one and the other upon body number two. The mutual force action between two bodies is called *action* in its effect upon the body which is being studied and *reaction* in its effect upon the body which is not being particularly studied, in the same way that a trade is called *buying* as it effects one person or *selling* as it affects the other person.

Inertia. — That property of a particle by virtue of which it perseveres in a state of rest or in a state of uniform motion in a straight line when not acted upon by an unbalanced force is called inertia; and the word inertia is generally extended in its meaning to include, not only this passive property, but also the idea of *reluctance to gain velocity.* Thus a given unbalanced force would have to act for a longer time on a body of large mass than upon a body of small mass to produce a given velocity, that is, the body of large mass has the greater reluctance to gain velocity, or a greater inertia in the extended sense of that term.

* An extremely fanciful statement of the second law of motion has crept into some elementary treatises on mechanics as follows: "The effect of a force is the same whether it act alone or in conjunction with other forces," meaning, of course, the accelerating effect. Now acceleration is proportional to force and *any effect which is proportional to a cause may be divided into parts and each part assigned as the effect of a corresponding part of the cause.* Thus, if the results of the labor of a number of men are proportional to the number of men, then it is justifiable from physical considerations to give each man an equal share of the profits; but if the results are *not* proportional to the number of men, it is *not* justifiable physically to give to each man an equal share of the profits. The principle of dividing cause and effect into parts which correspond each to each *when cause and effect are proportional,* is called the **principle of superposition** and it runs through the whole science of physics and chemistry.

33. Units of force. Formulation of the second law of motion. — (*a*) *Dynamic units of force.* — Having agreed to measure a force in terms of its effect in changing the velocity of a particle, we may choose as our unit of force that force which, acting as an unbalanced force on unit mass, will produce unit velocity in unit time (unit acceleration). Thus the *dyne* is that force which, acting for one second as an unbalanced force on a one-gram particle, will produce a velocity of one centimeter per second (an acceleration of one centimeter per second per second); and the *poundal* is that force which, acting for one second as an unbalanced force on a one-pound particle, will produce a velocity of one foot per second (an acceleration of one foot per second per second). The dyne is the c.g.s. unit of force and it is much used; the poundal is seldom used.

Having adopted as our unit of force that force which will produce unit acceleration of unit mass, it is evident, from Newton's second law, that F units of force will produce F units of acceleration of a particle of unit mass, or F/m units of acceleration of m units of mass; that is, F/m is *equal* to the acceleration a which the force F produces or $F/m = a$, whence

$$F = ma \tag{3}$$

in which F is the value of an unbalanced force in dynes (or poundals), m is the mass of a particle in grams (or pounds), and a is the acceleration in centimeters per second per second (or feet per second per second).

(*b*) *Weight. Gravitational units of force.* — Let g be the acceleration of a freely falling body produced by the unbalanced pull of the earth, let m be the mass of the body, and let W be the force, expressed in dynamics units, with which the earth pulls the body, then, according to equation (3) we have

$$W = mg \tag{4}$$

in which W is the weight of the body in dynes (or poundals), m is its mass in grams (or pounds), and g is the acceleration due to

gravity expressed in centimeters per second per second (or in feet per second per second). The value of g is about 980 centimeters per second per second (or 32 feet per second per second), so that, according to equation (4), the weight of a gram is about 980 dynes, and the weight of a pound is about 32 poundals.

The most convenient unit of force, for many purposes, is the force with which the earth pulls the unit mass. The force with which the earth pulls one gram is called the *gram-weight*, and the force with which the earth pulls the pound is called the *pound-weight*. Thus we may speak of 5,000 pounds-weight, meaning the force with which the earth pulls a mass of 5,000 pounds.

The slight variation in the value of the gram-weight and the pound-weight at different places on the earth is of no consequence in those cases where these units of force are used. Thus the tensile strength of a given grade of steel, repeatedly measured under conditions as nearly alike as it is possible to make them, will vary from, say, 100,000 to 105,000 pounds-weight per square inch, that is, the tensile strength of a given grade of steel is inherently indefinite (like the length of an angle worm!), and a variation of a few tenths of one per cent. in the value of the unit of force is of no consequence whatever.

It is often desirable, in discussing practical problems, to consider the relation between force, mass, and acceleration when forces are expressed in pounds-weight, mass in pounds, and acceleration in feet per second per second. In this case equation (3) becomes

$$F = \tfrac{1}{32} ma \tag{5}$$

inasmuch as the unit of force in this case, namely, one pound-weight, produces an acceleration of about 32 feet per second per second in a mass of one pound.

34. Measurement of force. — (*a*) *By the kinetic method.* — The force (unbalanced) acting on a body may be calculated by equation (3), the mass of the body being known and the acceleration being determined by observation. This method for measuring

force cannot be realized in its simplicity, but it forms the basis of many physical measurements.

(*b*) *By the counter-poise method.* — The strengths of materials are nearly always determined by applying, as the breaking force, the weight of a body or bodies of known mass, multiplied in a known ratio by a system of levers. The machine for carrying out such a test is called a *testing machine* and it is similar in many respects to the ordinary platform balance-scale.

(*c*) *By means of the spring scale.* — The spring scale is an arrangement in which an applied force stretches a spring and moves a pointer over a divided scale. The movement of the pointer is proportional to the force, and, the movement for a

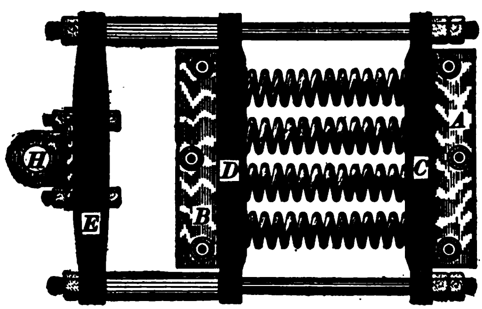


Fig. 25.

known force being observed, the scale can be divided so as to read the value of any force directly. The use of the spring-scale is exemplified in the measurement of the draw-bar pull of a locomotive. Figure 25 shows a scale designed for this purpose. The blocks AA are rigidly fixed to the "dynamometer car" and the link H couples with the locomotive. A pull on H moves the cross bar C and compresses the springs, and a push on H moves the cross bar D and compresses the springs. The relative motion of E and B actuates a pointer which plays over a divided scale.

UNIFORMLY ACCELERATED TRANSLATORY MOTION.

35. Falling bodies. — When a constant* unbalanced force acts upon a particle, the particle gains velocity at a constant rate. Such a particle is said to perform *uniformly accelerated motion.* A body falling freely under the action of the constant pull of the earth is, insofar as the friction of the air is negligible, an example of uniformly accelerated motion.

All bodies when falling freely gain velocity at the same rate, air friction being negligible. Thus two bricks together fall at exactly the same increasing speed as one brick alone. The doubled pull of the earth on the two bricks produces the same acceleration as the single pull of the earth on one brick. Doubling the force and doubling the mass leaves the acceleration unaltered.

Consider a particle which gains velocity at a constant rate of g centimeters per second per second, a falling body for example. The velocity gained in t seconds is

$$v = gt \tag{i}$$

Let v_1 be the initial velocity of the particle. Then $v_1 + gt$ is its velocity after t seconds, and its average† velocity during the t

* Constant in magnitude and unchanging in direction.

† Let the constantly increasing velocity of a falling body be represented by the ordinates of a curve of which the abscissas represent elapsed times. The "curve"

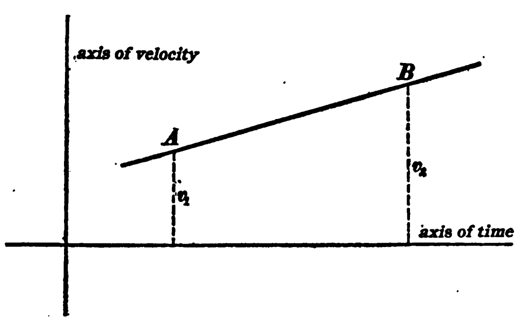


Fig. 26.

so plotted will be a straight line AB, Fig. 26, and the average ordinate of any portion AB of this line is equal to $\frac{1}{2}(v_1 + v_2)$.

seconds is $\frac{1}{2}[v_1 + (v_1 + gt)]$ or $v_1 + \frac{1}{2}gt$; and the distance d fallen by the particle during the t seconds is equal to the product of the average velocity into the time t. That is,

$$d = v_1 t + \tfrac{1}{2}gt^2 \tag{ii}$$

If v_1 is zero, equation (ii) becomes

$$d = \tfrac{1}{2}gt^2 \tag{iii}$$

Eliminating t between equations (i) and (iii), we have

$$v = \sqrt{2gd} \tag{iv}$$

which expresses the velocity of a body after it has fallen a distance d.

36. Projectiles. — When the initial velocity v_1 of a body is zero or when it is vertical, we have the ordinary case of a falling body, and equation (ii) of Art. 35 can be solved by simple arithmetic, the only complication being that v_1 is to be considered negative when it is upwards. When the initial velocity v_1 is not vertical, as in the case of a tossed ball, the falling body is called a projectile. In this case the entire argument of Art. 35 holds

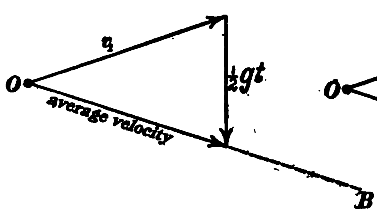


Fig. 27.

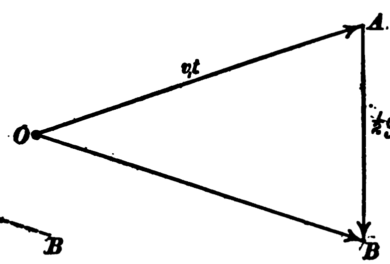


Fig. 28.

good but geometric addition must be substituted for arithmetic addition. Thus the average velocity of a projectile during t seconds is equal to the *geometric sum*, $v_1 + \frac{1}{2}gt$, as shown in Fig. 27, and after t seconds the projectile is in the line OB at a distance from O equal to t times the numerical value of the average velocity. Or one may find the position of the ball after t sec-

onds on the basis of equation (ii), considering that $v_1 t$ is a distance in the direction of v_1, that $\frac{1}{2}gt^2$ is a distance vertically downwards, and that the sum $v_1 t + \frac{1}{2}gt^2$ is a geometric addition as shown in Fig. 28.

The orbit of a projectile is a parabola. — This may be shown by choosing the x-axis of reference parallel to v_1 and the y-axis vertically downwards. Then $x = v_1 t$ and $y = \frac{1}{2}gt^2$, whence, by eliminating t we have the equation to the parabola.

The hodograph to the orbit of a projectile is a vertical straight line. — Draw the line OP, Fig. 29, representing the velocity of a projectile at a given instant, then, after t seconds, the vertical velocity gt will be gained, and the total velocity will be represented by OP'. Therefore, if we imagine the line OP to change so as to become OP' after t seconds and thus represent the changing velocity at each instant, then the end P will move vertically down wards at a constant velocity.

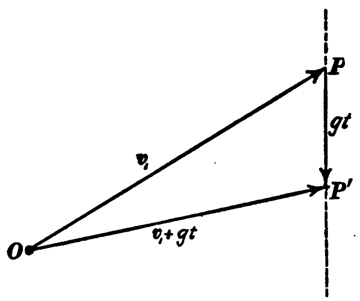


Fig. 29.

Range of a projectile. — The horizontal distance reached by a projectile when it comes to the level of the gun on its downward flight is called the *range* of the projectile. The range of a projectile, ignoring the effects of air friction, is given by the equation

$$l = \frac{2v_1^2 \sin\theta \cos\theta}{g}$$

in which v_1 is the initial velocity of the projectile, θ is the angle between the direction of v_1 and a horizontal line, and g is the acceleration of gravity. This expression for l may be easily derived with the help of the relations shown in Fig. 30, namely,

$$l = v_1 t \cos\theta$$

and

$$\tfrac{1}{2}gt^2 = v_1 t \sin\theta$$

37. Effect of air resistance on the motion of a projectile. — Bodies which are projected through the air do not have a constant downward acceleration, because of the resistance which the air offers to their motion, and therefore the simple theory of projectiles above outlined is not applicable in practice. The limitations of this simple theory may be stated in a general way as follows:

(*a*) In the first place the above simple theory is *not* limited to the motion of an ideal particle. The pull of the earth upon a projectile *tends only to produce translatory motion* and the effect

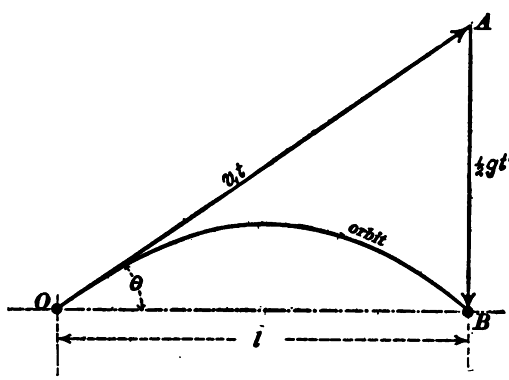


Fig. 30.

of the pull of the earth is the same whether the projected body is rotating or not, or whether the projected body is oscillating or not; *the center of mass of the body describes in every case a smooth parabolic curve* in accordance with the discussion of Art. 36. Thus, if an iron bar is thrown through the air, the center of mass of the bar describes a smooth parabolic orbit; or if the bar is projected by hitting it a sharp blow with a hammer, the center of mass of the bar describes a smooth parabolic orbit as before. This illustrates a very important extension of the idea of a material particle, namely, we may call *any body* a material particle, *whatever the character of its motion may be*, the idea being to direct one's attention solely to that part of the motion of the body which is translatory.

(*b*) In the case of a heavy body moving slowly, for example, an iron ball tossed from the hand, the resisting force of the air is very small compared with the weight of the body, and the motion of the body approximates very closely indeed to the ideal motion discussed in Art. 36.

(*c*) In the case of a light body, or in the case of a heavy body

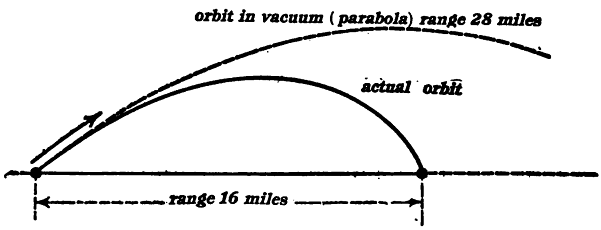


Fig. 31.

projected at high velocity, the resisting force of the air may be very large so that the motion of such a body differs widely from the ideal motion described in Art. 36. Thus, Fig. 31 shows the actual orbit of the heavy projectile from a modern high power

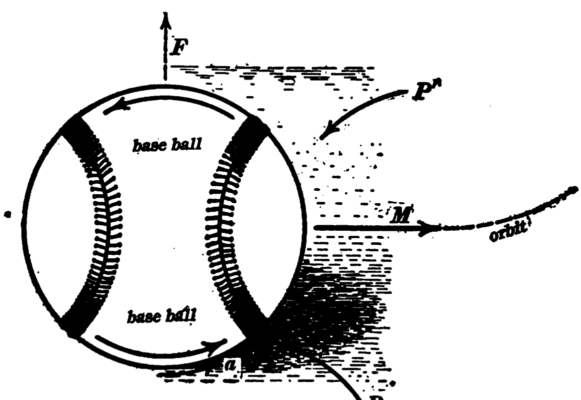


Fig. 32.

gun, and the dotted line shows what the orbit would be in a vacuum.

(*d*) The air friction on a rotating projectile generally gives rise to a force which pushes the projectile sidewise. This side force

is the cause of the curiously curved orbit of a "split-shot" tennis ball, and of a base ball pitched by an expert pitcher. The curved arrows, Fig. 32, show the direction of rotation of a base ball, the arrow M shows the direction of its translatory motion, the arrow F shows the side force above mentioned, and the dotted curve shows the curved orbit. The cause of the side force is that the very rapid motion of the rough surface of the ball at a produces an accumulation of air at P, whereas there is a less accumulation of air at P', and the excess of air at P causes the ball to glance to one side, thus curving the orbit.

TRANSLATORY MOTION IN A CIRCLE.

38. Velocity and acceleration of a particle moving steadily in a circular orbit. — Consider a particle which makes, steadily, n revolutions per second in a circular orbit of radius r. The circumference of the orbit is $2\pi r$, and, inasmuch as the particle traverses the circumference n times per second, its velocity v is

$$v = 2\pi rn \tag{6}$$

The magnitude, or numerical value, of the velocity v is constant; but its direction is changing continuously, this continual change of direction of v involves acceleration, and the state of affairs at each instant during the steady motion of a particle in a circular orbit, is most clearly shown by the use of the idea of the hodograph as explained in Art. 21. It is instructive, however, to discuss the motion of a particle in a circular orbit without explicit reference to the hodograph, as follows:

To determine the acceleration of a particle which is moving steadily in a circular orbit, it is necessary to consider the change of velocity during a very short interval of time. The circle, Fig. 33, represents the orbit of the particle, and at a given instant the particle is at P. At this instant the velocity v_1 of the particle is at right angles to PO and it is represented by the line $O'P'$ which is drawn from the fixed point O'. After the small lapse of time Δt, the particle will have moved a distance $v \cdot \Delta t$ to the point

Q, and its velocity will be v_2, which is represented by the line $O'Q'$. The change of velocity Δv is evidently parallel to PO (or to QO, for it must be remembered that the time interval Δt is infinitely small), and, since the triangles OPQ and $O'P'Q'$ are similar, we have

$$\frac{\Delta v}{v} = \frac{v \cdot \Delta t}{r} \tag{i}$$

in which v is written for the common numerical value of v_1 and v_2, and $v \cdot \Delta t$ is the length of the infinitesimal arc PQ which is

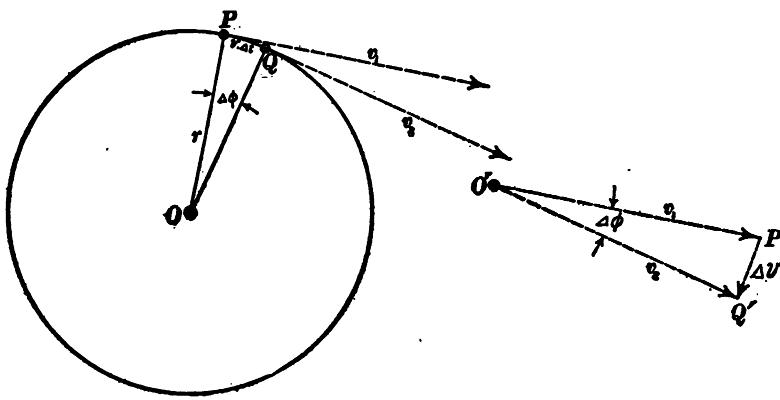


Fig. 33.

traversed by the particle during the time interval Δt. From equation (i) we have

$$\frac{\Delta v}{\Delta t} = \frac{v^2}{r} \tag{ii}$$

but, the change of velocity Δv divided by the time interval Δt during which the change takes place is the acceleration, so that, writing a for $\Delta v/\Delta t$, equation (ii) becomes

$$a = \frac{v^2}{r} \tag{7}$$

The direction of a is, of course, parallel to Δv, and Δv is parallel to PO. Therefore *a particle which moves steadily in a*

circular orbit of radius r has a steady acceleration towards the center of the circle, and this acceleration is equal to v^2/r, *where* v *is the steady velocity of the particle.*

It is sometimes convenient to have a expressed in terms of r and n, thus we may substitute the value of v from equation (6) into equation (7) and we have

$$a = 4\pi^2 n^2 r \tag{8}$$

Force required to constrain a particle to a circular orbit.—When a piece of metal is tied to a string and twirled in a circular orbit the string pulls steadily on the piece of metal, this pull of the string is an unbalanced force since no other force * acts on the piece of metal, and the value of the force in dynamic units is equal to the product of the mass of the particle and its acceleration, according to equation (3). Therefore we may substitute the value of a from equation (7) or equation (8) in equation (3) giving

$$F = \frac{mv^2}{r} \dagger \tag{9}$$

and

$$F = 4\pi^2 n^2 rm \dagger \tag{10}$$

where F is the force in dynamic units required to constrain a particle of mass m to a circular orbit of radius r, v is the velocity of the particle, and n is the number of revolutions per second.

39. Examples of motion in a circle.—(*a*) A one pound piece of metal twirled five revolutions per second in a circle four feet in radius would, according to equation (10), require a force of 3,948 poundals or about 123 pounds-weight to constrain it to its orbit.

(*b*) Each particle of a rotating wheel must be acted upon by an unbalanced force to constrain the particle to its circular path. If we consider only the rim of the wheel, neglecting the effect of

* Resistance of the air and force of gravity are here ignored.

† These equations express F in dynamic units, dynes or poundals as the case may be. If F is to be expressed in pounds-weight these equations become $F = \frac{1}{32} mv^2/r$ and $F = \frac{1}{32}(4\pi^2 n^2 rm)$, where m is the mass in pounds of the moving particle, v is its velocity in feet per second, r is the radius of the circle in feet, and n is the number of revolutions per second.

6

the spokes, it is evident that the necessity of the unbalanced radial forces gives rise to a state of tension in the rim. The tension in a barrel hoop presses each portion of the hoop radially against the barrel staves, and the outward push of the staves balances the radial force due to the tension of the hoop; but the tension in the rim of a rotating wheel produces an unbalanced radial force on each particle of the rim which force produces the radial acceleration which is associated with the circular motion.

(*c*) The tension of a belt produces a radial force which presses the belt radially against the face of the pulley. When the belt and pulley are in motion, however, a portion of the belt tension produces the radial forces required to constrain the particles of

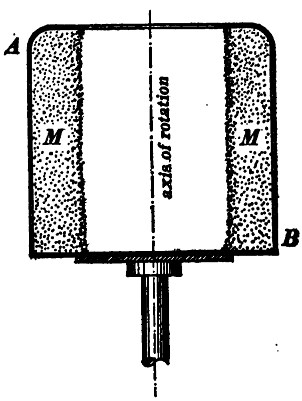


Fig. 34.

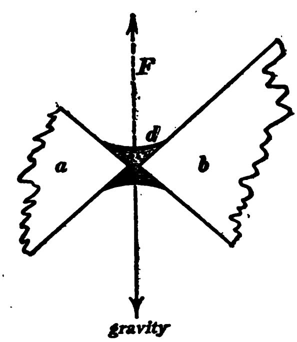


Fig. 35.

the belt to their circular paths; the portion of the belt tension so used is proportional to the square of the velocity of the belt and inversely proportional to the radius of the pulley ($a = v^2/r$). Therefore, belts running at high speeds on small pulleys have a troublesome tendency to slip, unless the tension is very great.

(*d*) The centrifugal drier which is used in laundries and in sugar refineries, is a rotating bowl *AB*, Fig. 34, with perforated sides, in which the material *MM* to be dried is placed. The action of the centrifugal drier may be clearly understood as follows: Consider two solid particles *a* and *b*, Fig. 35, with a drop

of water d clinging to them. Gravity of course pulls on the drop and the drop adheres to a and b so that the particles are able to exert on the drop a force F sufficient to balance gravity. In the centrifugal drier, however, the particles would have to exert upon the drop a force equal to $4\pi^2n^2rm$, where r is the radius of the circular path described by the particles a and b, m is the mass of the drop, and n is the speed of the drier bowl in revolutions per second, and this force $4\pi^2n^2rm$ may be, say, 1000 times as great as the weight of the drop; but the drop does not have sufficient adherence to the particles to enable the particles to hold to it with so great a force, and the result is that the drop is *not* constrained to the circular path, but flies off tangentially. The action of the centrifugal drier is as if a piece of wet cloth were jerked so quickly to one side as to leave the water behind.

(e) A locomotive on a railway curve describes a circular path and an unbalanced horizontal force (equal to mv^2/r) must push the locomotive towards the center of the curve in order that the locomotive may follow the curve, and of course this horizontal force must be exerted on the locomotive by the track. It is desirable, however, that the total force with which the track pushes on the locomotive (which is equal and opposite to the force with which the locomotive pushes on the track) shall be perpendicular to the plane of the track, and, therefore, the outside rail is always raised on a railway curve.

Let us consider the proper elevation to be given to the outside rail when the velocity v of the locomotive and the radius r of the curve are given. The rails are shown at a and b in Fig. 36, F is the total force that must act upon the locomotive, and the angle θ is the required elevation.

The vertical component of F is what sustains the locomotive against gravity. Therefore this vertical component is equal to mg where m is the mass of the locomotive and g is the acceleration of gravity. That is:

$$F \cos \theta = mg \tag{i}$$

The horizontal component of F is the unbalanced force which constrains the locomotive to its circular path. Therefore this horizontal component is equal to v^2m/r according to equation (9). That is:

$$F \sin\theta = \frac{v^2m}{r} \tag{ii}$$

Therefore, dividing equation (ii) by equation (i), member by member, we have

$$\tan\theta = \frac{v^2}{rg} \tag{iii}$$

When a locomotive is traveling on a curve it is evident that the whole locomotive is *rotating* about a vertical axis at such an angular speed that if the curve were a complete circle the loco-

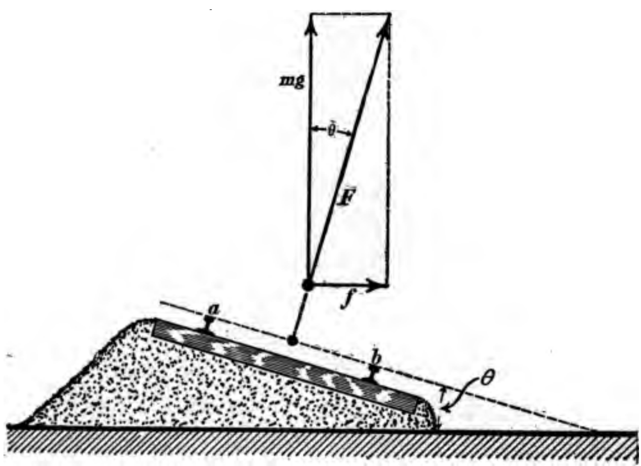


Fig. 36.

motive would make one rotation about a vertical axis every time it traversed the circular curve. Therefore a locomotive traveling on a curve does not perform pure translatory motion; but here again is an instance where the translatory motion may be considered by itself, for as long as the locomotive is *on* the curve, its rotating motion is *constant* and introduces no complication.

If a locomotive suddenly enters a curve from a straight portion of track, however, then the locomotive would have to change suddenly from zero rotatory motion to that speed of rotatory motion which corresponds to the curve, and the rails would have to exert an excessively great horizontal force on the front wheels of the locomotive. It is for this reason necessary to avoid sudden changes of curvature of a railway track. Thus Fig. 37 shows

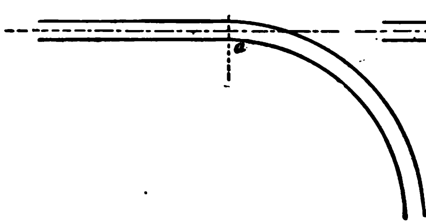


Fig. 37.

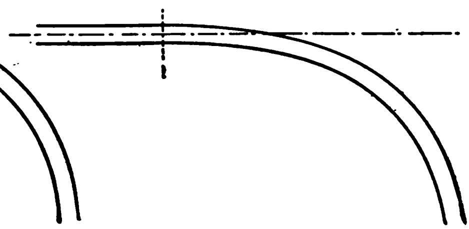


Fig. 38.

a straight portion of a track changing abruptly to a circular curve at the point a, and Fig. 38 shows the same straight portion changing gradually into the same circular curve.

40. The rotating hoop. — It is pointed out in the above examples of circular motion that the radial forces which constrain the particles of the rim of a rotating wheel to their circular paths are (ignoring effect of spokes) due to a state of tension in the rim. The tension of the rim is the force F with which any portion of the rim pulls on a contiguous portion. Let the circles, Fig. 39, represent a hoop of a radius r rotating n revolutions per second about the axis C, let the *mass per unit length* of the rim of the hoop be m', and let f be the unbalanced force pulling radially inwards on each unit length of the rim (due to the tension in the rim). Consider a very short portion of the rim of length $r \cdot \Delta\phi$, which subtends the angle $\Delta\phi$ as shown. The unbalanced force pulling the portion $r \cdot \Delta\phi$ radially inwards is $f \times r \cdot \Delta\phi$. Let F_2 be the force with which portion k pulls on the given portion $r \cdot \Delta\phi$ at the point a, and let F_1 be the force with which the portion j pulls on the given portion $r \cdot \Delta\phi$ at the point b. The forces F_1 and F_2 are equal, numerically, and they are tangent to

the circle at a and b respectively. Draw the two lines F_1 and F_2, Fig. 40, parallel to F_1 and F_2, Fig. 39, and complete the parallelogram of which F_1 and F_2 are the sides. Then the diagonal,

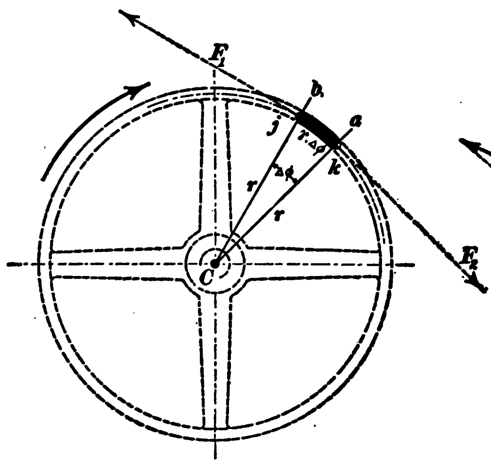


Fig. 39.

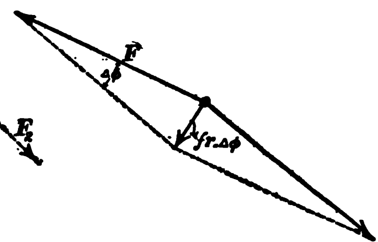


Fig. 40.

$f \times r \cdot \Delta\phi$, of this parallelogram represents the total unbalanced force which acts on the given portion $r \cdot \Delta\phi$ of the rim, and from the similar triangles of Fig. 39 and Fig. 40 we have

$$\frac{r \cdot \Delta\phi}{r} = \frac{fr \cdot \Delta\phi}{F}$$

in which F is written for the common numerical value of F_1 and F_2. Therefore

$$f = \frac{F}{r} \tag{i}$$

That is, the unbalanced inward pull on each unit length of a hoop is equal to the tension of the hoop divided by its radius.

Now the mass of unit length of the rim is equal to m'; and, according to equation (10), the force which must be pulling radially inwards on unit length of the rim to constrain it to its circular path, is equal to $4\pi^2 n^2 r$ times its mass m'. Therefore

writing $4\pi^2n^2rm'$ for f in equation (i) we have

$$F = 4\pi^2n^2r^2m' \tag{11}$$

in which F is the tension in dynes in a rim r centimeters in radius, rotating n revolutions per second, and m' is the mass in grams of one centimeter of the rim. If m' is expressed in pounds mass per foot of rim, r in feet, n in revolutions per second and F in pounds-weight, then

$$F = \tfrac{1}{32}(4\pi^2n^2r^2m')$$

Example. — The rim of a large flywheel has a mass of 250 pounds per foot, the radius of the wheel is 15 feet, the wheel rotates one revolution per second, and the tension of the rim (neglecting the effect of the spokes) is 69,350 pounds-weight.

TRANSLATORY HARMONIC MOTION.

41. Definition of harmonic motion. Utility of the idea. — Simple harmonic motion is the projection on a fixed straight line of uniform motion in a circle. Consider a point P', Fig. 41, moving uniformly around a circle of radius r at a speed of n revolutions per second, *the point P, which is the projection of P' on the line CD, performs simple harmonic motion.*

Vibration or cycle. — One complete up-and-down movement of the point P, Fig. 41, is called a *vibration* or a *cycle.*

Frequency. — The number of vibrations, or cycles, per second is called the *frequency* of the oscillations of the point P; this is, of course, equal to the number of revolutions per second of the point P'.

Period. The time required for the particle to complete one whole vibration, or cycle, is called the *period* of the harmonic motion. The relation between the frequency n and the period τ is obviously

$$n = \frac{1}{\tau} \tag{12}$$

Equilibrium position. When the vibrating particle P, Fig. 41, is at the point O, no force acts upon it, as explained below; the

point O is therefore called the *equilibrium position* of the vibrating particle.

Amplitude. The maximum distance from O reached by the vibrating particle is called the amplitude of its oscillations. This amplitude is equal to the radius r of the circle in Fig. 41.

Phase difference. Consider two points P' and Q', Fig. 42, both making n revolutions per second around the circle so that

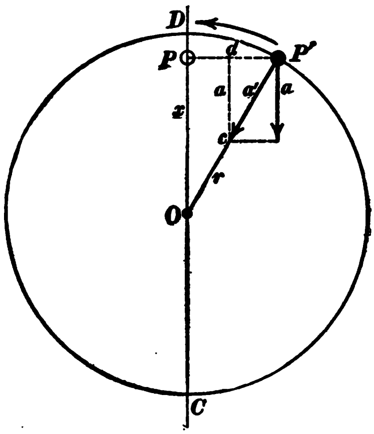


Fig. 41.

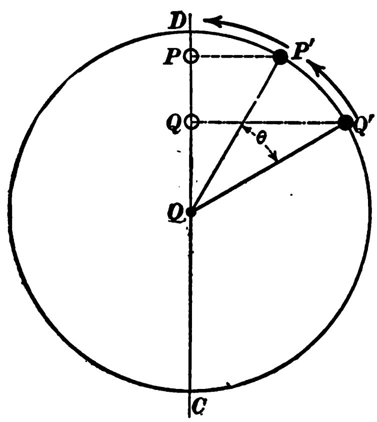


Fig. 42.

the angle θ is constant. The two oscillating particles P and Q are then said to *differ in phase* and the angle θ is called their *phase difference.*

The ideas involved in the peculiar type of motion which is performed by the particle P, in Fig. 41, are used throughout the study of oscillatory motion and wave motion. Thus the prongs of a vibrating tuning fork perform simple harmonic motion; the motion of a pendulum bob is, approximately, simple harmonic motion; when a rod, or a beam, or a bridge oscillates in the simplest possible manner, each particle of the rod, or beam, or bridge performs simple harmonic motion; when wave-motion of the simplest kind spreads through a body each particle of the body performs simple harmonic motion.

An example of simple harmonic motion in which all of the details of Fig. 41 are reproduced, is the motion of the cross-head of a steam engine with a long connecting rod. The crank pin moves at sensibly uniform speed in a circle, one component, only, of this motion is transmitted to the cross-head by the long connecting rod, and the cross-head moves to and fro in the manner of the point P in Fig. 41.

42. Acceleration of a particle in harmonic motion. The velocity of the point P in Fig. 41 is the vertical component of the velocity of the point P', and the acceleration a of the point P is the vertical component of the acceleration a', therefore, from the similar triangles $P'OP$ and $P'cd$ of Fig. 41 we have

$$\frac{a}{a'} = \frac{x}{r}$$

where x is the distance OP, and, since $a' = 4\pi^2n^2r$, according to equation (8), we have

$$a = -4\pi^2n^2x. \tag{13}$$

The minus sign is introduced for the reason that a is downwards (negative) when x is upwards (positive).

The force which must act on the particle P, Fig. 41, to cause it to move in the prescribed manner, is at each instant equal to ma, according to equation (3), therefore

$$F = -4\pi^2n^2mx \tag{14}$$

in which m is the mass of the oscillating particle, x is the distance of the particle from its equilibrium position at a given instant, n is the frequency of the oscillations, and F is the force which must act on the particle at the given instant.

The quantities n and m in equation (14) are constant. Therefore equation (14) indicates that *the force F which must at each instant act on a particle in harmonic motion is proportional to the distance x of the particle from its equilibrium position,* that is, we may write

$$F = -kx \tag{15}$$

where

$$k = 4\pi^2 n^2 m \tag{16}$$

or using $1/\tau$ for n, where τ is the period of one complete oscillation, we have

$$k = \frac{4\pi^2 m}{\tau^2}. \tag{17}$$

43. Examples of the application of equations (15), (16) and (17). — (*a*) *Application to a weight attached to the end of a flat spring.* A weight of mass m is fixed to one end of a flat steel spring S, the other end of which is clamped in a vise as shown in Fig. 43. If the weight M is pushed to one side through a

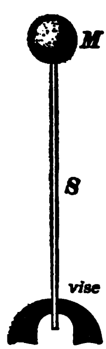


Fig. 43.

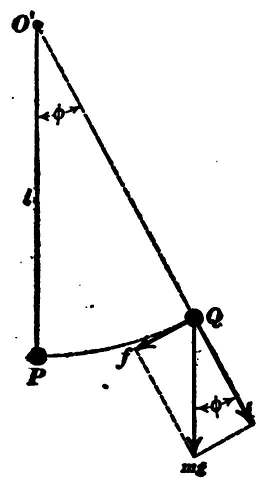


Fig. 44.

distance x, the spring exerts a force F which urges the weight back towards its equilibrium position *and this force is proportional to x.* Therefore, we may write

$$F = -kx \tag{i}$$

in which k is a constant, the value of which may be determined by observing the force required to *hold* the weight at a measured distance x from its equilibrium position.

Now since the equation (i) is identical to equation (15) it is evident that the weight, once started, will perform simple harmonic motion.

(*b*) *Application to the simple pendulum.* — The simple pendulum is an ideal pendulum consisting of a particle P, Fig. 44, suspended from a fixed point O by a string l of which the mass is negligible. If the particle P is moved to one side and released it will oscillate back and forth. *It is desired to show that these oscillations are simple harmonic oscillations, and that the period of one complete oscillation is equal to* $2\pi\sqrt{l/g}$, *where* g *is the acceleration of gravity.*

Let Q be the position of the oscillating particle at a given instant. The length x of the circular arc PQ is equal to $l\phi$, and the component Qf of the force mg with which gravity pulls downwards on the particle, is equal to $mg \sin \phi$ or, *if the angle* ϕ *is very small,* this force is equal to $mg \cdot \phi$, or to mg/l *times* ϕl, or to mg/l *times* x. Therefore, remembering that the force $Qf(=F)$ is to the left when the arc $PQ(=x)$ is to the right, we have

$$F = -\frac{mg}{l} \cdot x.$$

But this equation is identical in form to equation (15) since mg/l is constant, therefore the pendulum bob in Fig. 44 performs simple harmonic motion, and the equation expressing the period of the oscillations may be found by substituting mg/l for k in equation (17). In this way we find

$$\frac{mg}{l} = \frac{4\pi^2 m}{\tau^2}$$

or

$$\tau = 2\pi\sqrt{\frac{l}{g}}. \tag{18}$$

44. Harmonic motion represented by a curve of sines. — If the point P', Fig. 45, moves around the circle at uniform speed, the angle $P'OA$ is proportional to elapsed time, and it may be writ-

ten ωt where ω is a constant and t is elapsed time reckoned from the instant that P' was at A. Therefore we may write

$$x = r \sin \omega t. \tag{19}$$

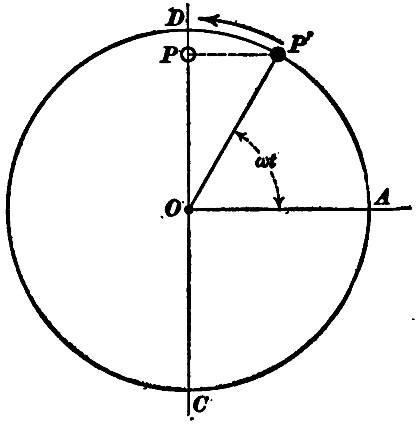


Fig. 45.

That is, the distance of the vibrating particle P from its equilibrium position O is proportional to the sine of a uniformly increasing angle, and if values of x be plotted as ordinates and the corresponding values of t (or ωt) as abscissas, we will have a curve of sines as shown in Fig. 46. If a fine pointer be attached to the prong of a tuning fork, the pointer may be made to trace a curve of sines by setting the fork in vibration and drawing the pointer uniformly across a piece of smoked glass.

SYSTEMS OF PARTICLES.

45. System of particles. The ideas of translatory motion may *conceivably* be extended so as to serve as a basis for the description of any motion of any body or substance, by looking upon the body or substance as a collection of particles and considering the varying position, velocity, and acceleration of each particle. A collection of particles treated in this way is called a *system of particles* or simply a *system.* Thus a rotating wheel is a system of particles, a portion of flowing water is a system of particles, a given amount of a gas is a system of particles. The word *configuration* is used when we wish to refer to the relative positions of the particles of a system; thus the configuration of a system is said to change when the particles change their relative positions.

A *closed system* is a system no particle of which has any force

acting on it from outside the system. There is no such thing in nature as a closed system, but the conception is useful nevertheless.

The cases in which it is not only conceivably possible, but actually feasible to study more or less complicated types of motion by treating the moving substance as a system of particles, are as follows:

(*a*) The case in which the system consists of very few bodies

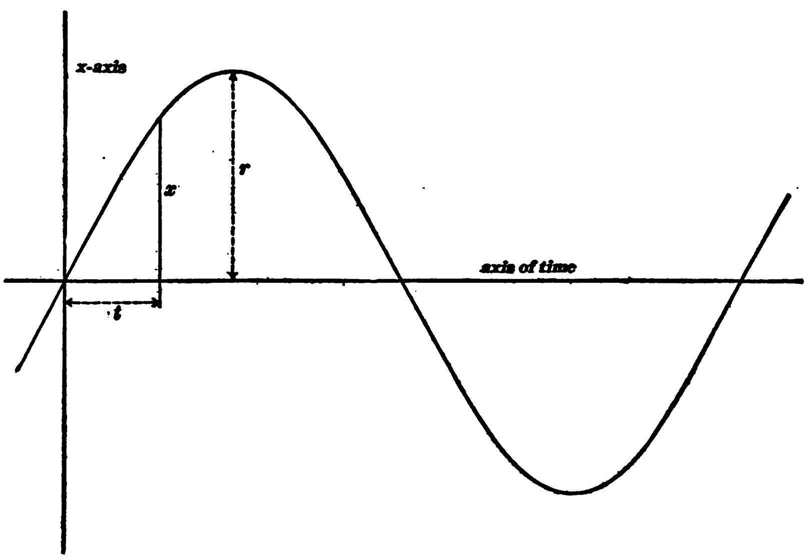


Fig. 46.

and where each body may be treated as a particle.* This case is exemplified by the sun and planets.

(*b*) The case in which the particles of a system move in a perfectly regular or orderly way. Thus the particles of a rotating wheel move in an orderly fashion, the particles in a smoothly flowing liquid move in an orderly fashion, the particles of a vibrating string move in an orderly fashion, the connected parts of any machine such as a steam engine or a printing press

* Or where each body is a *connected system*, see (*b*).

move in an orderly fashion. Any system in which orderly motion takes place is called a *connected system.*

(c) The case in which the particles of a system move in utter disorder, without any connection whatever with each other. In this case it would evidently be impossible to consider the actual motion of each particle, in fact the only possible treatment of such a system is a treatment based on the idea of *averages* and *probable departures therefrom.* Thus the very important kinetic theory of gases has been built up on the hypothesis that a gas consists of innumerable disconnected particles in disordered motion.

46. Momentum. — In the discussion of a system, the product mv of the mass of a particle and its velocity is of sufficient importance to warrant its receiving a name; it is called the *momentum* of the particle, it is a vector and its direction is the same as the velocity v of the particle.

When an unbalanced force acts upon a particle, of course the momentum of the particle changes; *the rate of change of the momentum is equal and parallel to the force.* This is evident when we consider that a change of velocity Δv means a *change of momentum equal to* $m \cdot \Delta v$, which, divided by the elapsed time Δt, gives the rate of change of momentum; but $\Delta v / \Delta t$ is equal to acceleration, so that $m \cdot \Delta v / \Delta t$ is equal to mass times acceleration, and this is equal to the unbalanced force, according to equation (3), Art. 33.

The mutual force-action of two particles cannot change the total momentum of the two particles. This is evident when we consider that the mutual force-action of two particles consists of two equal and opposite forces (action and reaction), so that while one particle gains momentum in one direction, the other particle gains momentum in the opposite direction at the same rate. The constancy of the total momentum of two particles, insofar as their mutual force-action is concerned, is called the *principle of the conservation of momentum.*

The principle of the conservation of momentum applies to any number of particles

insofar as their mutual force-actions are concerned. The total momentum of the particles of a system is never changed by the mutual force-action within the system, or, in other words, the total momentum of a closed system is constant.

47. Impact. — Consider two particles of which the masses are m_1 and m_2, and the velocities v_1 and v_2, respectively. The combined momentum of the two particles is $m_1v_1 + m_2v_2$. If the bodies collide, their velocities may change. Let V_1 and V_2 be the respective velocities after impact. Then $m_1V_1 + m_2V_2$ is the total momentum of the bodies after impact, and by the principle of the conservation of momentum, we have

$$m_1v_1 + m_2v_2 = m_1V_1 + m_2V_2 \tag{i}$$

Impact of inelastic balls. — When an inelastic ball, such as a ball of soft clay strikes squarely against another, the two balls move after impact as a single body so that V_1 and V_2 are equal, and this common velocity after impact is completely determined by equation (i).

Impact of perfectly elastic balls. — Consider two elastic balls moving at velocities v_1 and v_2 in the same straight line (v_1 and v_2 being opposite in sign if the balls are moving in opposite directions). Let the masses of the balls be m_1 and m_2 respectively.

When these balls collide they are distorted, and at a certain instant the distortion reaches a maximum, after which the balls rebound from each other and the distortion is relieved. *When the distortion of the two balls has reached its maximum, the two balls are at the instant moving at common velocity c, which is determined by the equation*

$$(m_1 + m_2)c = m_1v_1 + m_2v_2 \tag{ii}$$

During the time that the balls are being distorted, which time we shall call the first half of the impact, the first ball loses* an amount of velocity $(v_1 - c)$ and the second ball loses* an amount of velocity $(v_2 - c)$. During the time that the balls are being relieved from distortion, which time we shall call the second half of the impact, they are assumed to act on each other with precisely the same series of forces as during the first half of the impact, only in reverse order. This is what is meant by the assumption that the two balls are perfectly elastic. Therefore during the second half of the impact, each ball loses the same amount of velocity as it lost during the first half of the impact, that is, the total loss of velocity by the first ball is $2(v_1 - c)$ and the total loss of velocity by the second ball is $2(v_2 - c)$, so that

$$V_1 = v_1 - 2(v_1 - c)$$

and

$$V_2 = v_2 - 2(v_2 - c)$$

or

$$V_1 = 2c - v_1 \tag{iii}$$

and

$$V_2 = 2c - v_2 \tag{iv}$$

in which V_1 and V_2 are the respective velocities of the balls after impact.

* The velocity c lies between v_1 and v_2 so that if $(v_1 - c)$ is positive then $(v_2 - c)$ must be negative.

Substituting the value of c from equation (ii) in equations (iii) and (iv) we have

$$V_1 = \frac{m_1v_1 + 2m_2v_2 - m_2v_1}{m_1 + m_2} \tag{v}$$

$$V_2 = \frac{2m_1v_1 + m_2v_2 - m_1v_2}{m_1 + m_2} \tag{vi}$$

The simplest case is where $m_1 = m_2$ and where $v_2 = 0$, that is where the balls are similar, and where the first ball only is in motion before impact. In this case the result may be derived from equations (v) and (vi) but it is more instructive to derive the result anew. The common velocity c at the middle of the impact is equal to $\frac{1}{2}v_1$. That is, the first ball loses half its velocity and the second ball gains an equal amount of velocity during the first half of the impact. During the second half of the impact the first ball loses the remainder of its velocity and comes to a standstill, and the second ball gains once more an equal amount of velocity so that its velocity is now equal to the initial velocity v_1 of the first ball. That is, when an elastic ball A strikes squarely against a similar stationary ball B, the ball A stops, and the ball B moves on with the full original velocity of A. If A is heavier than B, then both balls move in the same direction after the impact. If B is heavier than A, then A moves backwards, or has a negative velocity after the impact.

48. Motion of the center of mass of a system. — The center of mass of a system has been defined in physical terms in Art. 25. The center of mass of a body of uniform density is at the geometrical center of the body. The center of mass of two particles lies on the line joining them, and its distance from each particle is inversely proportional to the mass of the particle. Thus the center of mass of the earth and moon is on the line joining the center of the earth and the center of the moon, and it is about 80 times as far from the center of the moon as it is from the center of the earth (3,000 miles from the center of the earth), inasmuch as the mass of the earth is about 80 times as great as the mass of the moon.

The center of mass of a system remains stationary, or continues to move with uniform velocity in a straight line, if the vector sum of all of the forces which act on the system is zero.

For example, consider an emery wheel mounted on a shaft and rotating at high speed. If the center of mass of the wheel lies in the axis of the shaft, it of course remains stationary as the wheel rotates, and the only force that need be exerted on the shaft by the bearings is the steady upward force required to balance

the downward pull of the earth on the wheel. A rotating machine part is said to be *balanced* when its center of mass is in its axis of rotation.

When the center of mass of a system is not stationary, and does not move with uniform velocity in a straight line, then the vector sum F_s of the forces which act on the system is not zero.

In fact, the acceleration A of the center of mass of a system of particles, the vector sum F_s of the forces which act on the system, and the total mass M of the system are related to each other precisely in the same way, as the acceleration, force, and mass of a single particle. That is, as fully explained in Art. 50, we have

$$F_s = MA \tag{20}$$

Example 1. — Consider an emery wheel of which the center of mass lies at a distance r to one side of the axis of rotation, then, as the wheel rotates, the center of mass describes a circular path of radius r, the acceleration of the center of mass is equal to $4\pi^2n^2r$ at each instant, and a side force equal to $4\pi^2n^2rM$ and *parallel to r at each instant* must act on the axle to constrain the center of mass to its circular path, precisely as if the entire mass of the wheel were concentrated at its center of mass.

Example 2. — The centrifugal drier consists of a rapidly rotating bowl mounted on top of a vertical spindle, and the materials to be dried are placed in this bowl. It is impossible to keep the bowl and contents even approximately balanced, so that, if the spindle were carried in a rigid bearing, the machine would be disabled in a short time because of the very great forces that would be brought into play in constraining the center of mass of bowl and contents to move in a circular path. This difficulty is obviated by supporting the spindle at the lower end only, in a long bearing mounted on springs to hold it approximately vertical. The bowl, contents, and spindle then rotate about a line passing through their center of mass and through the center of the flexible bearing, and, although the bowl and spindle seem to wobble badly (inasmuch as they do not rotate about the axis of figure),

nevertheless the machine runs quite smoothly, producing but little vibration in the supporting frame.

Example 3. — If two balls, which are tied together with a short string, are thrown in such a way that the string is kept stretched while the balls revolve rapidly about one another, a certain point of the string will describe a smooth parabolic curve, just as a simple projectile would do. This point of the string is the center of mass of the two balls. The center of mass of the earth and the moon describes an elliptic orbit about the sun once a year, while the earth and moon rotate about their center of mass once every lunar month, in a manner very similar to the motion of the two balls just described.

49. The balancing of a rotating machine part. Any part of a machine which is to rotate rapidly must be adjusted so that its center of mass lies in the axis of rotation.* This adjustment is called *balancing*, and a machine part so adjusted is said to be balanced. A machine part which is to be balanced, a dynamo armature for example, is mounted on its shaft and the ends of the shaft are placed upon two straight level rails. If the center of mass is in the axis of the shaft, the whole will stand in equilibrium in any position; whereas, if the center of mass is not in the axis of the shaft, the whole will come to rest with the center of mass at the lowest possible position, and material must be removed from one side until the center of mass is in the center of the shaft.

Figure 47 shows a wheel mounted on a pair of balancing rails. Such a pair of balancing rails is a prominent feature in a shop where the fly-wheels of large engines have to be balanced.

50. Equations of center of mass. The position of the center of mass of a system of particles may be expressed in terms of the positions and masses of all of the particles in the system as follows: Let x be the x-coordinate of a particle whose mass is m, let x' be the x-coordinate of a particle whose mass is m', let x'' be the x-coordi-

* A machine part which is long in the direction of the shaft upon which it rotates, may have its center of mass in the axis of the shaft and yet the bearings may have to exert constraining forces upon the shaft as the part rotates. A long cylinder loaded on opposite sides at the two ends is an example.

nate of a particle whose mass is m'' and so on, then the sum $mx + m'x' + m''x'' +$ etc., divided by the total mass of the system, namely, $m + m' + m'' +$ etc., gives the x-coordinate of the center of mass of the system. That is, the x-coordinate of the center of mass is

$$X = \frac{\Sigma mx}{\Sigma m} \tag{21}$$

and exactly similar expressions may be formulated for the y-coordinate and for the z-coordinate of the center of mass.

If the origin of coordinates is at the center of mass of the system then, of course, X is equal to zero, and equation (21) becomes

$$\Sigma mx = 0. \tag{22}$$

In order to show that equation (20) is true, it is sufficient to consider only the x-component of A, and the x-components of the accelerations of the respective particles.

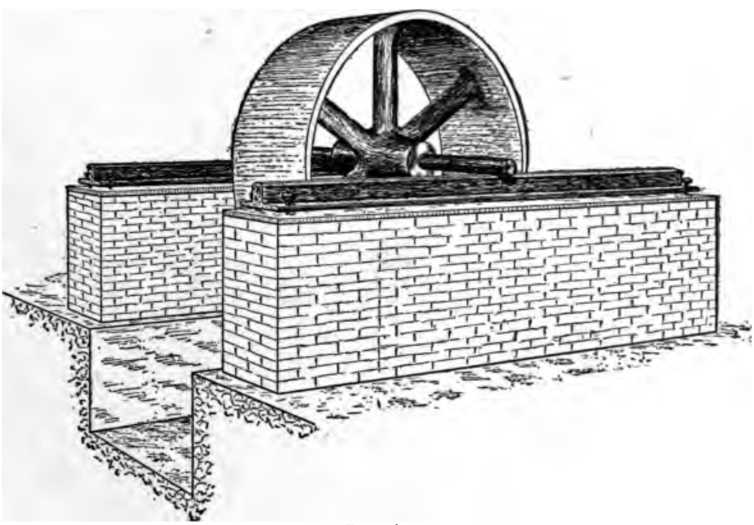


Fig. 47.

The x-component of A is d^2X/dt^2 and the x-components of the accelerations of the respective particles are d^2x/dt^2, d^2x'/dt^2, d^2x''/dt^2 and so on. Therefore, writing M for Σm in equation (21), and differentiating twice with respect to time, we have

$$M\frac{d^2X}{dt^2} = m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} + \frac{d^2x''}{dt^2} + \text{etc.},$$

but $m(d^2x/dt^2)$ is the x-component of the force acting on the particle m, $m'(d^2x'/dt^2)$ is the x-component of the force acting on the particle m' and so on, so that the right-

hand member of this equation is the sum of the x-components of all the forces acting on the particles of the system, and this is equal to the sum of the x-components of all of the *external* forces acting on the particles of the system inasmuch as mutual force-actions between the particles of the system cancel out of this sum because such mutual force-actions consist of pairs of equal and opposite forces. Therefore, the right hand member of the above equation is the x-component of the total external force F_e which acts on the system and the above equation reduces to

$$M \text{ times } x\text{-component of } A = x\text{-component of } F_e \qquad \text{(i)}$$

and we may show in exactly the same way that

$$M \text{ times } y\text{-component of } A = y\text{-component of } F_e \qquad \text{(ii)}$$

and

$$M \text{ times } z\text{-component of } A = z\text{-component of } F_e \qquad \text{(iii)}$$

These three equations are equivalent exactly to the single vector equation (20).

Problems.

45. A train having a mass of 350 tons (2,000 pounds) starting from rest reaches a speed of 50 miles per hour in 2½ minutes. What is the average pull of the locomotive during 2½ minutes, dragging forces of friction being neglected?

46. The above train moving at a speed of 50 miles per hour is brought to a standstill in 16 seconds by the brakes. What is the average retarding force in pounds-weight due to the brakes?

47. An elevator reaches full speed of 8 feet per second 2½ seconds after starting. With what average force in pounds-weight does a 160-pound man push down on the floor while the elevator is starting up? The elevator is stopped (when moving up at full speed) in 1½ seconds. With what average force in pounds-weight does a 160-pound man push down on the floor while the elevator is stopping?

Note. — In the first case the upward push of the floor on the man exceeds the weight of the man by the amount which is necessary to produce the upward acceleration; in the second case the weight of the man exceeds the upward push of the floor by the amount which is necessary to produce the downward acceleration.

48. An elevator car has a mass of 1,000 pounds. It gains a velocity upwards of 8 feet per second in 2½ seconds after starting from rest. Calculate (*a*) the tension on the rope while the car is stationary, (*b*) the average tension of the rope while the

car is starting upward, and (*c*) the tension of the rope while the car is moving at the full speed of 8 feet per second.

49. A train having a mass of 1,200 tons (2,000 pounds) is to be accelerated at ½ mile per hour per second up a ½ per cent. grade. The train friction is 10 pounds per ton. Find the necessary draw-bar pull of the locomotive.

Note. — A ½ per cent. grade is one that rises 1 foot in 200 feet of horizontal distance.

50. A cord is strung over a pulley. At one end of the cord is a 10 pound weight, and at the other end of the cord is a 11 pound weight. Neglecting the weight of the cord and the friction and mass of the pulley, find the acceleration of each weight and the tension of the cord.

51. A falling ball passes a given point at a velocity of 12 feet per second. How far below the point is the ball after 5 seconds? How far does the ball fall during the fifth second after passing the given point?

52. A heavy iron ball is tossed at a velocity of 20 feet per second in a direction 30° above the horizontal. What are its horizontal and vertical distances from the starting point after ¾ second?

Note. — Find vertical and horizontal components of the initial velocity. The latter component remains unchanged while the vertical motion of the ball is precisely what it would be if it had no horizontal motion.

53. A heavy shot is thrown in a direction 30° above the horizontal, it strikes the ground 50 feet from the thrower, and the shot is 5½ feet above the ground when it leaves the thrower's hand. What is the initial velocity v of the shot?

Note. — The horizontal velocity $v \cos 30°$ is constant, the time of flight is $t = 100 \text{ feet} \div (v \cos 30°)$, and 5 feet is equal to $v \sin 30° \times t + \frac{1}{2}gt^2$.

54. An 80-ton (2,000 pounds) locomotive goes round a railway curve of which the radius is 600 feet at a velocity of 65 feet per second. With what force in pounds-weight do the flanges of the wheels of the locomotive push against the outer rail when the outer rail is not elevated?

55. Calculate the proper elevation to be given to the outer rail on a railway curve of 600 feet radius for a train speed of 65 feet per second, the width of the track being 4 feet 8½ inches.

56*a*. The tension of a belt is 50 pounds-weight. With what force in pounds-weight does the belt push against each inch of circumference of a pulley 12 inches in diameter when the pulley is stationary?

Note. — The static relation between tension in a circular hoop and actual outward forces acting on each part of the hoop is the same as the relation between tension and the unbalanced inward forces in the case of a rotating hoop as discussed in Art. 40. See Art. 26 on D'Alembert's principle.

56*b*. The mass of each inch of length of the belt specified in problem 56 is ¼ pound. With what force in pounds-weight does the belt push against each inch of circumference of the 12-inch pulley when the pulley revolves at a speed of 1,500 revolutions per minute, the actual tension of the belt being 50 pounds-weight?

57. The car next to the locomotive in a train is 35 feet long between bumpers and it is pulled at each end with a force of 10,000 pounds (the force at the rear end of the car is of course somewhat less than the force at the front end). The train rounds a circular curve of 1,000 feet radius at a speed of 20 miles per hour. The car with its load weighs 100,000 pounds. Find the horizontal force, at right angles to the track, with which the track acts on the car.

Note.—The portion of a train directly behind the locomotive is under tension like a belt, and the tension helps to constrain the cars to their circular path exactly as in the case of a belt passing around a pulley. In solving this problem it is sufficiently accurate to use the formula $F = T/r$ in which T is the tension ot a belt, r is the radius of the circular arc formed by the belt, and F is the radial force per unit length of belt due to T.

58. A force of 5×10^6 dynes deflects the end of the spring in Fig. 43 through a distance of 1.25 centimeters. What is the value of the constant k in equation (15), and in terms of what unit is this constant expressed? How much force would be required to deflect the end of the spring through a distance of 2 centimeters?

59. A mass of 2 kilograms is attached to the end of the spring specified in problem 58, and the mass is set vibrating. How many complete vibrations will it make per minute?

60. A force of 10 pounds-weight deflects the end of the spring in Fig. 43 through a distance of 0.02 foot. What is the value of the constant k in equation (15) and in terms of what unit is this constant expressed? A mass of 10 pounds is attached to the end of the spring, how many complete vibrations will the 10 pound mass make per minute?

61. What is the length l of a simple pendulum which makes one complete vibration per second at a place where the acceleration of gravity is 981 centimeters per second per second?

62. A wheel has a mass of 50 pounds, its center of mass is 0.2 inch from the axis of the shaft upon which the wheel rotates, and the speed of the wheel is 600 revolutions per minute. How much force in pounds-weight must act on the shaft to constrain the center of mass to its circular path? What is the direction of the force at each instant?

63. A ballistic pendulum AB, Fig. 63p, is suspended by two cords ss, the length of each of which is 400 centimeters, and the body AB weighs 10 pounds. A rifle bullet of which the mass is 0.005 pound, strikes AB at the point indicated by the short arrow, and the velocity imparted to AB carries it through a horizontal distance of 8 inches before it is brought to rest by gravity. Find the velocity of the bullet. The acceleration of gravity is 32 feet per second per second.

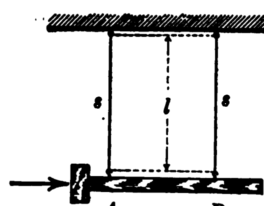


Fig. 63p.

Note.—The center of mass of AB describes the arc of a circle of which the radius is l. Calculate the vertical movement of AB from the known value of l and the specified horizontal movement of AB. Then calculate the velocity of AB which would suffice to lift AB through this vertical distance, and then calculate the velocity of the bullet by using the principle of the conservation of momentum.

64. A ball weighing 550 pounds is shot from a 150,000-pound

gun at a velocity of 2,500 feet per second. What is the backward velocity of the gun as the ball leaves the muzzle? Suppose the gun is allowed to move back two feet during the recoil, what is the average value of the force required to bring it to rest?

65. An ivory ball of which the mass is 500 grams, and of which the velocity is 100 centimeters per second, collides with a stationary ivory ball of which the mass is 1,000 grams, the line connecting the centers of the balls being parallel to the velocity of the moving ball. Find the common velocity of both balls after their relative motion has been reduced to zero during the first half of the impact, and find the velocity of each ball after impact; specify direction of each velocity.

Note. — Assume that the ivory balls are perfectly elastic as explained in Art. 47.

CHAPTER VI.

FRICTION. WORK AND ENERGY.

51. Friction.—A body in motion is always acted upon by dragging forces which oppose its motion and tend to bring it to rest. This action is called *friction.*

Sliding friction.—When one body slides on another the motion is opposed by a frictional drag. Thus the cross-head of a steam engine slides back and forth on the guides, a rotating shaft slides in its bearings, and the motion is in each case opposed by a frictional drag.

Fluid friction.—The flow of water through a pipe or channel, the motion of a boat, and the motion of a projectile through the air are opposed by friction. This type of friction is called fluid friction and it is discussed in a subsequent chapter.

Rolling friction.—The frictional drag upon a wheeled vehicle is due in part to the sliding friction at the journals, in part to the friction of the air, and in part to the continual yielding of the road or track under the wheels. The effect of this yielding is very much as if the vehicle were continually going up a hill, the top of which is never reached. The frictional drag on a wheeled vehicle due to the yielding of the road or track is sometimes called rolling friction.

Frictional drag due to unevenness of a road bed.—When a vehicle is drawn very slowly over a rough road, the wheels roll "up hill," as it were, when they strike a small stone and then "down hill" again when they leave the stone, and the *average* value of the pull required to draw the vehicle is not effected by unevenness of road bed; but if the speed of the vehicle is increased, the unevenness of the road bed produces a very considerable frictional drag, the effect is as if the wheels were being all the time "rolled up" a succession of small hills not to "roll

down" again, but to come down each time with a bump. This kind of friction shows itself in the vibration and swaying of a vehicle, and it is one of the most prominent causes of frictional drag upon a vehicle which is driven at high speed.

52. Coefficient of sliding friction. — The horizontal force H required to cause a body to slide steadily over the smooth horizontal surface of another body is approximately proportional to the vertical force V which pushes the body against the surface. That is

$$H = \mu V \tag{23}$$

in which V is the force with which a body is pushed against any smooth surface, and H is the force, parallel to the surface, which causes the body to slide. The proportionality factor μ is called the *coefficient of friction;* it is nearly independent of the contact area of the sliding substances and it does not vary greatly with the velocity of sliding. Thus the coefficient of friction of wood on a smooth metal surface is about 0.40, the coefficient of friction of smooth brass on smooth steel (not oiled) is about 0.22.

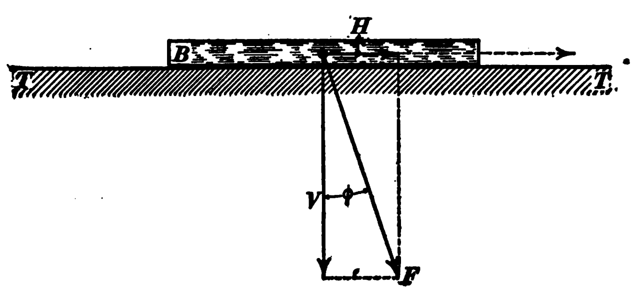


Fig. 48.

Angle of friction. Consider a block B, Fig. 48, sliding on a table TT in the direction of the dotted arrow, and let V be the force with which the block is pushed against the table and H the force necessary to keep the block in motion. Then, since the ratio H/V is constant (that is, if V is large, H is large in proportion), it is evident that the angle ϕ between V and the result-

ant force F is constant. This angle is called the *angle of friction* of the given substances B and T, and, evidently, the tangent of ϕ is equal to the coefficient of friction, μ, of the sliding substances.

It is important to notice that the force F in Fig. 48 is the total force which the sliding block exerts upon the table. *The block can exert on the table any force whatever, the direction of which lies inside of a cone described by rotating the line F about V as an axis, but the block cannot exert on the table a force the direction of which lies outside of this cone.* This statement assumes that a force H which is sufficient to keep the block B sliding is sufficient to start it sliding. In fact, a slightly greater force is required to start the block (not on account of acceleration, but because of sticking) than is required to keep it sliding.

Nature of sliding friction. — The friction between two sliding surfaces is, no doubt, due in part to a continual interlocking and release of fine protuberances on the sliding surfaces, and in part to a continual welding together and tearing apart of the substances as they come into intimate contact. When the sliding surfaces are distinctly rough there is no regularity whatever in the friction. Surfaces which are fairly smooth, however, have a well defined coefficient of friction, especially if they are made of unlike materials. Thus wood sliding on metal, polished steel sliding on brass or babbitt metal, hard steel sliding upon the polished surface of a jewel, all have fairly well defined coefficients of friction. The coefficient of friction is generally small in value for hard polished dissimilar substances.

Similar substances usually have a large coefficient of friction and frequently the friction is very irregular between similar substances. Thus brass on brass tends to weld and tear in a most remarkable manner, and a clean plate of glass cannot be made to slide on another clean glass plate at all (if the surfaces are very clean) unless there is an air cushion between them.

53. Active forces and inactive forces. Definition of work. — Nothing is more completely established by experience than the

necessity of employing an active agent such as a horse or a steam engine to drive the machinery of a mill or factory, to draw a car, or to propel a boat; and although the immediate purpose of the driving force may be described in each case by saying that the driving force overcomes or balances the opposing forces of friction, still the fact remains that the operation of driving a machine or propelling a boat involves a continued effort or cost. Indeed to supply a man with the thing (energy) which will drive his mill or factory, is to supply him with a commodity as real as the wheat he grinds or the iron which he fabricates into articles of commerce. Wheat and iron are sharply defined as commodities in the popular mind on the basis of many generations of commercial activity, because wheat and iron can be stored up and taken from place to place, and because change of ownership is so easily accomplished and so simply accounted for. That which serves to drive a mill or factory, however, cannot be stored up except to a very limited extent, and it is only in recent years that means have been devised for transmitting it from place to place and that an exact system of accounting has been established for governing its exchange. A clear idea of energy does not exist as yet in the popular mind, and the following definitions cannot be expected to convey a full and clear idea at once.

The common feature of every case in which motion is maintained is that *a force is exerted upon a moving body and in the direction in which the body moves.* Such a force is called an *active force* *, and to keep up an active force requires continuous effort or cost.

A force which acts on a stationary body, on the other hand, may be kept up indefinitely, without cost or effort; and such a force is called an *inactive force.* Thus a weight resting on a table continues to push downward on the table, a weight suspended by a string continues to pull on the string, the main

*** An active force is any mutual force action between two bodies one of which moves with respect to the other. To push on the front door of a moving car is not to exert an active force.**

spring of a watch will continue indefinitely to exert a force upon the wheels of the watch if the watch is stopped.

The idea of an inactive force is applicable also to a force which acts on a moving body but at right angles to the direction in which the body moves. Thus the vertical pull of the earth on a railway train which moves along a level track is an inactive force, the forces with which the spokes of a wheel pull on the moving rim of the wheel are inactive forces.

An active force is said to do work, and the amount of work done in any given time is equal to the product of the force and the distance that the body has moved in the direction of the force. That is

$$W = Fd \tag{24}$$

in which F is the force acting on a body, and W is the work done by the force during the time that the body moves a distance d in the direction of F. If d is not parallel to F, then $W = Fd \cos \theta$, where θ is the angle between F and d.

Units of work. The unit of work is the work done by unit force during the time that the body upon which the force acts moves through unit distance parallel to the force.

The *erg*, which is the *c. g. s.* unit of work, is the work done by a force of one dyne during the time that the body upon which the force acts moves through a distance of one centimeter in the direction of the force. The erg is, for most purposes, inconveniently small, and a multiple of this unit, the *joule*, is much used in practice. The *joule* * is equal to ten million ergs, (10^7 ergs).

The work done by a force of one pound-weight during the time that the body upon which the force acts moves through a distance of one foot in the direction of the force, is called the foot-pound.†

* It is frequently convenient to have a name for that unit of force which multiplied by one centimeter gives one joule of work, according to equation (24). This unit of force may be called the joule per centimeter.

† The *kilogram-meter* is the work done by a force of one kilogram-weight during the time that the body upon which the force acts moves through a distance of one

54. Power. — The rate at which an agent does work is called the *power* of that agent. Thus a locomotive exerts a pull of 15,000 pounds-weight on a train and draws the train through a distance of 500 feet in 10 seconds. The work done is 7,500,000 foot-pounds which, divided by the time interval of ten seconds, gives 750,000 foot-pounds per second as the rate at which the locomotive does work.

Units of power. Power may, of course, be expressed in ergs per second, in joules per second, or in foot-pounds per second. The unit of power, one joule per second, is called a *watt.* The *horse-power,* which is extensively used by engineers, is equal to 746 watts or to 550 foot-pounds per second.

Power developed by an active force. — Consider a force F acting upon a body which moves in the direction of the force at velocity v. During t seconds the body moves through the distance vt and the amount of work done is $F \times vt$ according to equation (24), and, dividing this amount of work by the time, we have

$$P = Fv \tag{25}$$

in which P is the power developed by an active force F, and v is the velocity with which the body, upon which F acts, moves in the direction of F. If F is expressed in dynes and v in centimeters per second, then P is expressed in ergs per second; if F is expressed in pounds-weight and v in feet per second, then P is expressed in foot-pounds per second.

Example. — A horse pulls with a force of 200 pounds weight in drawing a loaded cart at a velocity of 3 feet per second and develops 600 foot-pounds per second of power.

Measurement of power. — Nearly all practical measurements relating to work are measurements of power. The power of an agent may be measured as follows:

(*a*) The value of an active force and the velocity of the body

meter in the direction of the force. The foot-pound unit of work is used quite generally by American and English engineers, and the kilogram-meter unit of work is used in those countries where the metric system has been adopted.

upon which the force acts, may be measured and the power may then be calculated according to equation (25).

Examples. — (1) The draw-bar pull of a passenger locomotive is measured by means of a heavy spring scale and found to be 6,000 pounds, and the velocity of the locomotive, as determined by the distance traveled in a given time, is found to be 90 feet per second. From these data the net power developed by the locomotive (not counting the power required to propel the locomotive itself) is found to be 540,000 foot-pounds per second, or 991 horse-power.

(2) Let a be the area in square inches of the piston of a steam engine, let p be the average steam pressure in the cylinder in pounds per square inch as measured by a steam-engine indicator, let l be the length of stroke of the piston in feet, and let n be the number of revolutions per second made by the engine. Then the average force pushing on the piston is pa pounds-weight, and the work done during a single stroke is $p \times a \times l$ foot-pounds, and since the number of single strokes per second is $2n$, the power developed by the steam is $pal \times 2n$, or $2paln$ foot-pounds per second. The power of an engine determined in this way is called its *indicated power*, to distinguish it from the power delivered by the engine to the machinery which it drives. The power delivered by an engine is always less than its indicated power on account of frictional losses in the engine.

(3) An engine to be tested is loaded by applying a brake to its flywheel; the pull on the brake (reduced to the circumference of the flywheel) is 200 pounds-weight; the velocity of the circumference of the flywheel, as determined from the measured diameter of the wheel and its observed speed in revolutions per second, is 80 feet per second; and the power developed by the engine is equal to 200 pounds $\times$ 80 feet per second, which is equal to 16,000 foot-pounds per second or 29 horse-power. The power of an engine determined in this way is called its *brake power*.

Figure 49 shows the arrangement of a brake for measuring the power of an engine, or of any agent, like an electric motor or

water wheel, which delivers power from a pulley. The spring scale S measures the force at the end of the brake arm, and this observed force is multiplied by a/r to find the equivalent force at the surface of the pulley, where a is the length of the arm as shown in Fig. 49 and r is the radius of the pulley.

(*b*) Power is frequently measured electrically. Thus the power in watts delivered by a direct-current dynamo is equal to the product of the electromotive force of the dynamo in volts by the current in amperes delivered by the dynamo.

Power-time units of work. — Inasmuch as nearly all practical

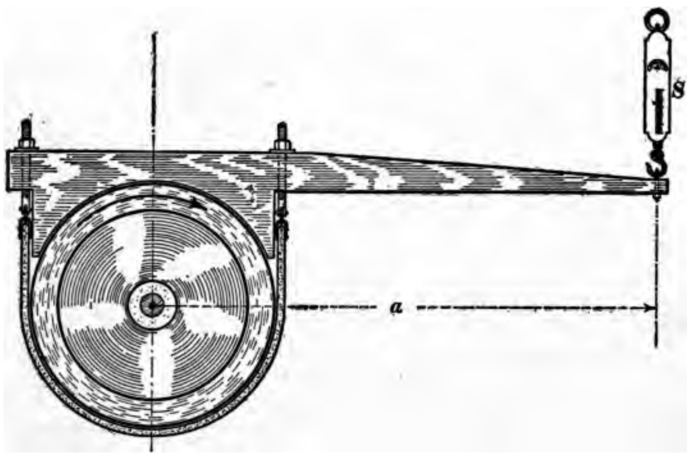


Fig. 49.

measurements relating to work are measurements of power, it has come about that a given amount of work is often expressed as the product of power and time. The *watt-hour* is the amount of work done in one hour by an agent which does work at the rate of one watt, the *kilowatt-hour* is the amount of work done in one hour by an agent which does work at the rate of one kilowatt (one kilowatt is 1,000 watts), and the *horse-power-hour* is the amount of work done in one hour by an agent which does work at the rate of one horse-power.

Efficiency. — The efficiency of a machine, like a water wheel,

a steam engine, a dynamo, or a motor, which transforms energy, is defined as the ratio of the power developed by the machine to the total power delivered to the machine.

ENERGY.

55. Definition of energy. Limits of the present discussion. — Any agent which is able to do work is said to possess *energy*, and the amount of energy an agent possesses is equal to the total work the agent can do. Thus the spring of a clock when it is wound up is in a condition to do a definite amount of work and it is therefore said to possess a definite amount of energy.

In developing the idea of energy it is important to distinguish between an agent which merely transforms energy and an agent which actually has within itself the ability to do a certain amount of work. Thus the steam engine merely transforms the energy of fuel into mechanical work, and a water wheel merely transforms the energy of an elevated store of water into mechanical work, whereas a clock spring when wound up has a store of energy within itself.

Whenever a substance or a system of substances gives up energy which it has in store, the substance or system of substances always undergoes change. Thus the fuel which supplies the energy to a steam engine and the food which supplies the energy to a horse, undergo *chemical change;* the steam which carries the energy of the fuel from the boiler to the engine *cools off* or undergoes a *thermal change* when it gives up its energy to the engine; a clock spring *changes its shape* as it gives off energy; an elevated store of water *changes its position* as it gives off energy; the heavy fly wheel of a steam engine does the work of the engine for a few moments after the steam is shut off and the fly wheel *changes its velocity* as it gives off its energy.

Not only does a substance undergo a change when it *gives up energy* by *doing work*, but a substance which *receives energy* or has *work done upon it* undergoes a change. Thus when air is compressed by a bicycle pump, work is done on the air and it

becomes warm; the work done in keeping up the motion of any machine or device produces heat at the places where friction occurs; when a clock spring is wound up it stores energy by its change of shape; when water is pumped into an elevated tank it stores energy by its change of position; a large part of the work which is expended on a heavy railway train at starting is stored in the train by its change of velocity.

We are now facing a very important question; shall we attempt a complete discussion of the whole theory of energy at once by examining into all kinds of changes which take place when a substance does work or has work done upon it? or shall we base our discussion on one thing at a time? Most assuredly the latter. Therefore let us proceed to discuss the energy relations involved in purely mechanical changes, namely, changes of position, changes of velocity, and changes of shape,* and let us exclude everything else from our present discussion such as chemical changes and thermal changes.

In attempting to exclude thermal changes from our present discussion, however, we are confronted by the fact that friction (with its accompanying thermal changes) is always in evidence everywhere; and it requires a very high degree of analytical power to think only of purely mechanical changes in the face of such a fact. This necessary feat of mental effort is greatly facilitated by the use of the idea of a *frictionless system;* and this word will be used whenever it is desired to direct the reader's attention exclusively to the energy relations that are involved in purely mechanical changes.

Before proceeding to a minute examination into the mechanical theory of energy, it is desirable to establish the ideas of *kinetic energy* and *potential energy* on the basis of general experience. Suppose that a post, standing beside a railway track, is to be pulled out of the ground; can a car-load of stone be made to do the work? Certainly it can. All that is necessary is to have the car moving past the post and to throw over the post a loop

* Changes of shape are discussed in Chapter VIII.

of cable which is attached to the moving car. A moving car is able to do work; and when it does work its velocity is reduced, and its store of energy decreased. The energy which a body stores by virtue of its velocity is called the *kinetic energy* of the body.

It is also a familiar fact that a weight can drive a clock, but in doing so the position of the weight changes and its store of energy is reduced. The energy which a body stores by virtue of its position is called the *potential energy* of the body.

The physical reality which lies behind the terms kinetic energy and potential energy can perhaps be shown most clearly by considering a bicycle rider. Suppose that the rider faces a steep hill or a sandy stretch of road where he is called upon to do an unusual amount of work. Every bicycle rider realizes the advantage of having a large velocity in such an emergency. This *advantage of velocity* is called kinetic energy.* Or suppose that a bicycle rider wishes to use his whole strength, or more if he had it, in covering a certain distance; every bicycle rider realizes the advantage of being on top of a hill in such an emergency; this *advantage of position* is called potential energy.

56. Kinetic energy of a particle.—The kinetic energy of a particle is given by the equation

$$W = \tfrac{1}{2}mv^2 \tag{26}$$

in which W is the kinetic energy in ergs, m is the mass of the particle in grams, and v is its velocity in centimeters per second.

Proof of equation (26). The kinetic energy of a particle may not only be defined as the work it can do when stopped, but also as the work required to establish its motion. Let a constant unbalanced force F act upon a particle of mass m, then

$$F = ma. \tag{i}$$

* Of course a body can have velocity only in relation to another body and the idea of kinetic energy is an idea which applies strictly to a system of particles but not to an individual particle. The velocity in equation (26) is velocity referred to the earth.

After t seconds the velocity gained is

$$v = at \tag{ii}$$

and the distance traveled is

$$d = \tfrac{1}{2}at^2. \tag{iii}$$

Therefore, multiplying equations (i) and (iii), member by member, we have

$$Fd = \tfrac{1}{2}ma^2t^2, \tag{iv}$$

but Fd is equal to the work done on the particle and a^2t^2 is equal to v^2, according to equation (ii), so that equation (iv) reduces to $W = \frac{1}{2}mv^2$.

The kinetic energy of a system of particles is, of course, equal to the sum of the kinetic energies of the individual particles of the system.

When mass is expressed in pounds and velocity in feet per second, then the kinetic energy of a particle in foot-pounds is given, approximately, by the equation

$$W = \tfrac{1}{64}mv^2. \tag{27}$$

57. Potential energy. The energy stored in a system by virtue of the configuration of the system, that is, by virtue of the relative positions of the parts of the system, is called the *potential energy* of the system. For example a weight stores energy by virtue of its position relative to the earth; a bent spring stores energy by virtue of its elastic distortion.

It is impossible to assign a definite amount of potential energy to a system which has a given configuration, for it is impracticable to assign a definite limiting configuration beyond which the system cannot go. Thus the weight of a clock might have its *available* store of potential energy increased by boring a hole in the clock case so that the weight could move down to the floor, then a hole could be bored in the floor and eventually a deep well could be dug in the ground. In order to be able to speak definitely of the potential energy of a weight it is necessary, there-

fore, to assign an arbitrary *zero position* and to reckon the potential energy of any other given position as the work the weight can do in changing from the given position to the chosen zero position.

In general the potential energy of any system in a given configuration may be defined as the amount of work the system can do in changing from the given configuration to an arbitrarily chosen *zero configuration.*

Conservative systems. — A system (frictionless) which does the same amount of work in passing from one configuration to another, whatever the intermediate stages may be through which the system passes, is called a *conservative system*, **and the idea of potential energy applies only to such systems.** Suppose, for example, that a weight would do more work in moving from a given position to its chosen zero position over one path A than over another path B, see Fig. 50a; then the potential energy of the weight in the given position would be indefinite; and if the weight were carried around the closed path AB in the direction of the arrows, then a large amount of work would be done in passing down path A and only a *portion* of this work would be required to carry the weight back to the given position over path B. That is, work would be created every time the weight completed the cycle of motion around AB, and we would have "perpetual motion," that is a machine which could do work without suffering any permanent change of any kind.* *All physical systems are conservative insofar as purely mechanical changes are concerned;* and experience shows that all physical systems are conservative when changes

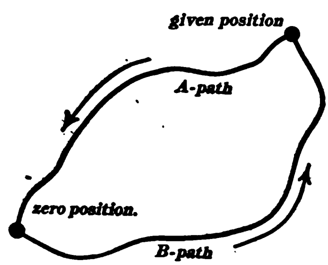


Fig. 50*a*.

* The idea involved int his discussion of Fig. 50a may be strengthened by introducing the idea of cheating to which it stands in clear apposition. Suppose one were to hold a weight in his hand and allow it to move downwards in full view of a class, and then bring it again to its former position by passing it behind his back where it is out of sight with the idea of avoiding the doing of work!

of all kinds, mechanical, chemical, thermal and electrical, are taken into account; that is, the energy that a system gives off when it undergoes any change whatever, depends only upon the initial and final states of the system, and is independent of the intermediate stages through which the system may be made to pass.

Perpetual motion impossible. — A perpetual motion machine would be a device which would furnish a continuous supply of energy for driving machinery. Most of the attempts to produce perpetual motion have been quite ridiculous, but on the other

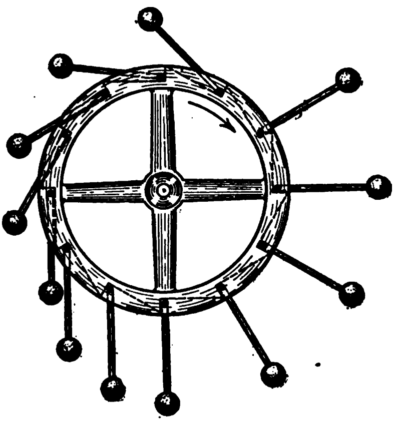


Fig. 50*b*.

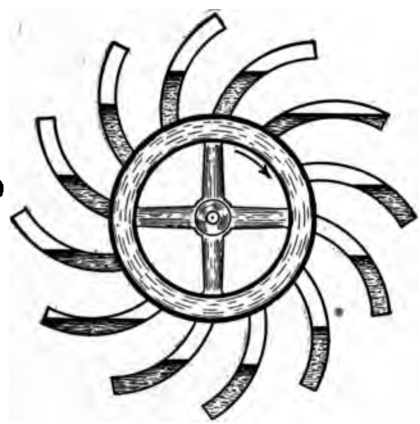


Fig. 50*c*.

hand many attempts have been quite reasonable. The reasonable attempts have nearly all been attempts to get more work out of a weight while it falls along one path than is required to carry the weight back to its starting point along another path. Figures 50*b* and 50*c* show two perpetual motion devices which were proposed and tried about 1750. Figure 50*b* is a rachet wheel to which a number of hinged arms are attached, each arm carrying a heavy weight. Figure 50*c* is a wheel to the rim of which a number of bent tubes are attached, each tube containing mercury. The arrows show the directions in which the wheels were expected to be driven by the increased leverage of the falling weights or of the falling mercury.

58. Mutual relation between the kinetic energy and the potential energy of a closed system. — A closed system is a system upon which no outside forces act, or, in other words, it is a system which neither gives off nor receives energy. Such a system does not exist in nature, but the most clearly intelligible statement of the idea of the impossibility of perpetual motion may be made by referring to such an ideal system.

Suppose that the particles of a closed system are in motion and let us consider what takes place in any very short interval of time. In the first place, each particle moves through a certain small distance, the configuration of the system is changed accordingly, and the potential energy of the system decreases by an amount which is equal to the total work done on all of the particles by their mutual force actions. In the second place, the kinetic energy of each particle increases by an amount equal to the work done upon it, and, of course, the total kinetic energy of the system increases by an amount which is equal to the total work done on all of the particles by their mutual force actions. Therefore, the decrease (or increase) of potential energy of a closed system is always equal to the accompanying increase (or decrease) of the kinetic energy of the system, or, in other words, the sum of the potential and kinetic energies of a closed system is constant.

59. The principle of the conservation of energy. — The argument of Art. 58 is based upon the simplest aspect of Newton's laws of motion, and the conclusion reached, namely, the constancy of the total energy of a closed system, follows directly from the idea of a conservative system as developed in Art. 57, that is, from the idea of potential energy. The principle of the conservation of energy reduced to its simplest terms is that **the work done by a system depends only upon the initial and final states of the system and it is hopeless to seek a roundabout method for bringing the system back to its initial state by a smaller expenditure of work.**

The usual statement of the principle of the conservation of

energy is that *energy can be neither created nor destroyed;* but this statement is so completely abstracted from actual physical considerations, that it is almost meaningless.

60. The application of the principle of work to statics. The principle of virtual work. — Consider a body which is acted upon by any number of forces; then, *if the body should be given any slight displacement whatever, the total work which would be done by all the forces is called the virtual work, and this work is equal to zero when the forces are in equilibrium.*

This principle of virtual work furnishes the simplest basis for formulating the conditions of equilibrium in many complicated mechanisms.

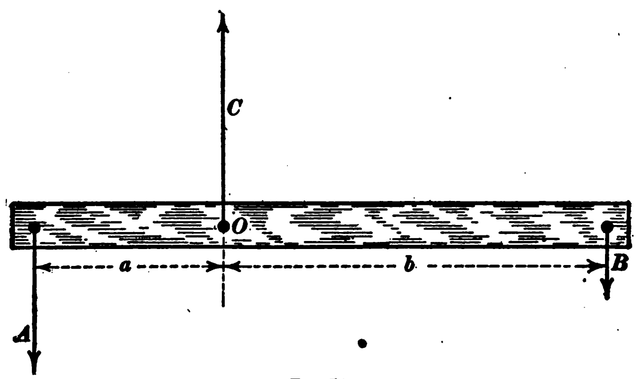


Fig. 51.

Example 1. Let the body shown in Fig. 51 be turned about the point O through the angle $\Delta\phi$ causing the point of application of the force A to move downwards through the distance $a \cdot \Delta\phi$ and the point of application of the force B to move upwards through the distance $b \cdot \Delta\phi$; then $Aa \cdot \Delta\phi$ is the work *done on* the body by the force A, and $Bb \cdot \Delta\phi$ is the work *done by* the body on the agent which exerts the force B. Therefore $Bb \cdot \Delta\phi$ is to be considered as negative work *done on* the given body, so that according to the above principle we have

$$Aa \cdot \Delta\phi + Bb \cdot \Delta\phi = 0$$

or

$$Aa + Bb = 0$$

which shows the relation between the two forces which must be satisfied if their combined tendency to turn the body about the point O is zero.

Example 2. The following discussion of the tension in a rotating hoop furnishes a good example of the use of D'Alembert's principle (see Art. 26) and of the principle of virtual work.

Let m' be the mass per unit length of circumference of a circular hoop of radius r rotating about its axis of figure at a speed of n revolutions per second. The radial acceleration of each particle of the hoop is $4\pi^2n^2r$ according to equation (8), so that each unit of length of circumference u of the hoop must be pulled inwards by an unbalanced force f equal to $4\pi^2n^2rm'$, according to equation (10) as shown in Fig. 52*a*.

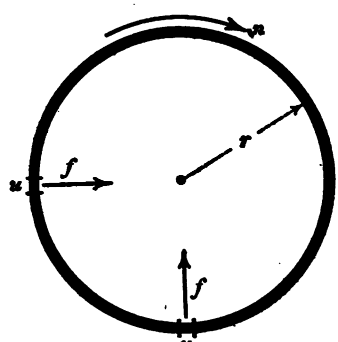


Fig. 52*a*.

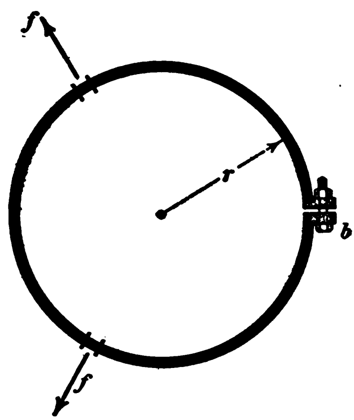


Fig. 52*b*.

By applying D'Alembert's principle this problem may be reduced to a problem in statics as follows: Given a stationary hoop of radius r on each unit length of circumference of which an outward force $f(= 4\pi^2n^2rm')$ acts, required the tension F of the hoop.

By applying the principle of virtual work the relation between the tension F and the outward force f per unit length of circumference may be determined as follows. Imagine the bolt b

to be shortened by a certain small amount l, all forces remaining unchanged in value; the work done in thus shortening the bolt would be Fl; but to shorten the bolt by the amount l would shorten the radius of the hoop by $l/2\pi$, so that each unit of the circumference would move inwards through the distance $l/2\pi$ against the force f, and, since there are $2\pi r$ units of circumference, or $2\pi r$ forces like f, the total work done by the contracting hoop would be $2\pi r \times f \times l/2\pi$ which is equal to rfl. The work which would be done in shortening the bolt, however, is the work that would be expended in contracting the hoop, so that $Fl = rfl$ or

$$F = rf$$

in which F is the tension of a hoop of radius r, and f is the outward force pushing on each unit of circumference of the hoop. If nothing pushes out on the hoop, however, then the tension F produces an unbalanced inward force equal to f, which unbalanced force suffices to constrain each particle of the hoop to its circular path, so that, writing $4\pi^2n^2rm'$ for f in the above equation, we have

$$F = 4\pi^2n^2r^2m'$$

which is identical to equation (11) of Art. 40.

PROBLEMS.

66. A 165-pound man climbs a height of 40 feet in 11 seconds. How much work is done, and at what rate? Express the work in foot-pounds, and in joules; and express the power in foot-pounds per second, in horse-power, and in watts.

67. A horse pulls upon a plow with a force of 100 pounds weight and travels 3 miles per hour. What power is developed? Express the result in foot-pounds per second, in watts, and in horse-power.

68. A belt traveling at a velocity of 70 feet per second transmits 360 horse-power. What is the difference in the tension of the belt between the tight and loose sides in pounds weight?

69. A stream furnishes 500 cubic feet of water per second at a head of 15 feet. What power can be developed from this stream by a water wheel of which the efficiency is 60%?

70. The engines of a steamship develop 20,000 horse-power, of which 30 per cent. is represented in the forward thrust of the screw in propelling the ship at a speed of 17 miles per hour. What is the forward thrust of the screw in pounds-weight?

71. An electric motor has an efficiency of 80 per cent. and electrical energy costs 5 cents per kilowatt-hour. How much does the output of the motor cost per horse-power hour?

72. A 1,000 horse-power boiler and engine plant costs about $70,000 complete, including land, building, boilers, engines and auxiliary apparatus such as pumps and feed water heaters. The cost of operating this plant continuously, night and day, is as follows:

Interest on investment	5 per cent. per annum.
Depreciation	10 " " " "
Maintenance and repairs	4 " " " "
Taxes and insurance	2 " " " "

Labor $30 per day, 365 days in the year.
Coal $2.00 per ton.

The average demand for power is 50 per cent. of the rated power output of the plant, that is 500 horse-power, and the consumption of coal is 2½ pounds per horse-power-hour. Find the cost of a horse-power-hour delivered by the engine. Ans. 0.83 cent.

73. The above engine will drive a 700 kilowatt dynamo, that is a dynamo capable of delivering 700 kilowatts. The cost of dynamo, station wiring and switch-board apparatus is $20,000. The average output of the dynamo is 350 kilowatts (corresponding to 500 horse-power output of engine). Calculate the cost of electrical energy per kilowatt-hour at the station, allowing 21 per cent. for interest, depreciation, etc., on the electrical machinery and allowing $5 per day additional for labor. Ans. 1.31 cents.

74. A steam engine indicator shows an average steam pressure

of 55 pounds per square inch (reckoned above atmospheric pressure) during each stroke of a steam engine, and the engine exhausts into a condenser where the pressure is 13 pounds per square inch below atmospheric pressure. The diameter of the piston is 16 inches, the diameter of the piston rod is 3 inches, the length of stroke is 24 inches, and the engine makes 75 revolutions per minute. Find the indicated horse-power of the engine.

75. A brake test of a steam engine gave the following data: speed of engine 200 revolutions per minute, length of brake arm (*a*, Fig. 49) $7\frac{1}{2}$ feet, observed force at end of brake arm and at right angles to arm 240 pounds-weight. Find the brake horse-power of the engine.

76. A fan blower is mounted on a cradle which swings on knife edges in the line of the axis of the fan. When the fan is driven, the cradle tends to tip to one side and this tendency is balanced by a weight sliding on a horizontal lever arm, as shown in Fig. 76*p*. The belt is thrown off the fan, the sliding weight moved to

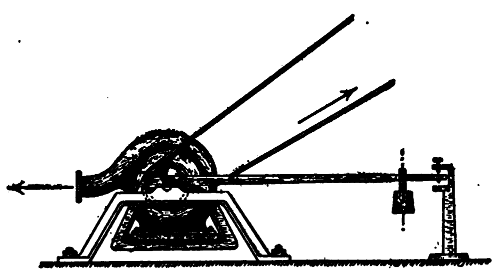


Fig. 76*p*.

give a balance and the "zero position" of the weight is observed. The fan is then driven at a speed of 1800 revolutions per minute and the weight (10 pounds) has to be moved $6\frac{3}{4}$ inches from its zero position to balance the driving torque exerted by the belt on the fan. Find the power expended in driving the fan and express it in horse-power and in watts.

77. Four idle pulleys *A*, *B*, *C* and *D*, Fig. 77*p*, are mounted in a frame which is free to rotate about the point *O* which is the

point of intersection of the left hand stretches of belt. A weight W slides along a lever arm which is fixed to the rocking frame so that the tilting action of the right hand stretches of belt may be balanced and measured. When no power is transmitted by the belt, the tension of the belt is the same everywhere and, under these conditions, the weight W is adjusted to its zero position to give a balance. When power is transmitted to a given machine the belt tensions F' and F'' differ and the weight W is moved away

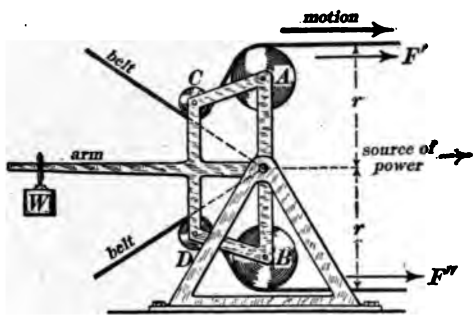


Fig. 77*p*.

from O to again give a balance. This movement of the weight W from its zero position is 16 inches, the mass of W is 55 pounds, the distance r is 24 inches, and the speed of the belt is 80 feet per second. Find the power transmitted by the belt.

78. A shaft transmits 100 horse-power and runs at a speed of 250 revolutions per minute. Calculate the torque exerted on the shaft. Express the result in pound-feet, in pound-inches, and in dyne-centimeters.

79. A steamship has a gross mass of 25,000 tons. What is the kinetic energy of the ship at a speed of 18 miles per hour? Express the result in foot-pounds and in horse-power hours.

80. A bicycle rider has a 50-foot hill to climb. What velocity must he have at starting to relieve him from the doing of one third of the work required?

81. The rim of the fly-wheel of a metal punch is 5 feet in diameter and its mass is 560 pounds. At what initial speed must

the fly-wheel run in order that the punch may exert a force of 72,000 pounds through a distance of one inch and reduce the speed of the fly-wheel only 30 per cent.?

82. A counterpoise of 3/4 pound balances a weight of 100 pounds wherever the weight may be placed on the platform of a balance scale. In what way and to what extent does the platform move when the counterpoise moves 1/2 inch downwards?

83. A screw-jack is turned by a lever of which the radial length is 18 inches, and the pitch of the screw is 3/8 inch. What is the lifting force produced by a pull of 100 pounds on the end of the lever, neglecting friction?

Note. — Consider the distance travelled by the end of the lever and the travel of the screw in one complete turn, and apply the principle of virtual work.

84. The differential pulley consists of a large pulley A and a smaller pulley B made in one piece, and a third pulley C all threaded with an endless chain as shown in Fig. 84*p*. The

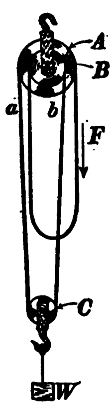


Fig. 84*p*.

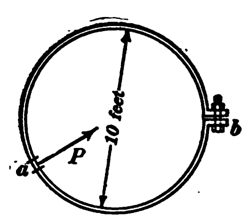


Fig. 85*p*.

pulleys A and B are sprocket wheels with notches which engage the links of the chain so that the chain cannot slip on A and B. One turn of A and B takes in at a a length of chain which is equal to the circumference of the larger pulley A and pays out at b a length of chain which is equal to the circumference of the smaller pulley B.

The circumference of A contains 12 notches, the circumference of B contains 11 notches and the length of each link of the chain is 1½ inches. What is the lifting force produced by a pull of 150 pounds at F, neglecting friction?

85. A steel hoop 10 feet in diameter is clamped around a large wooden tank by means of the bolt *b*, Fig. 85*p*. The tension in the strap is 1000 pounds-weight (= force exerted by the bolt), find the force exerted by each foot-length of hoop against the tank.

CHAPTER VII.

ROTATORY MOTION.

61. Rotation about a fixed axis. Definitions. — The simplest case of rotatory motion is that which is exemplified by the rotation of a wheel about a fixed axis. We shall first consider this simple case in detail and then proceed to the more complicated rotation about a moving axis. In order to rivet the attention to rotatory motion to the exclusion of movements of distortion, the idea of a *rigid body* will be used throughout the chapter, a rigid body being an ideal body which cannot change its shape or size.

Angular velocity. — Let ϕ be the angle turned by a rotating body during t seconds; the quotient ϕ/t is called the *average angular velocity* of the body during the t seconds. If the time interval is very short, the quotient $\Delta\phi/\Delta t$ is the actual angular velocity of the body at the given instant, $\Delta\phi$ being the angle turned by the rotating body during the short time interval Δt. When the angle ϕ is expressed in radians and time t in seconds, then the quotient ϕ/t is in *radians per second.* Angular velocity is expressed in radians per second throughout this chapter. In practice, angular velocity is generally expressed in revolutions per second. There are 2π radians in one revolution, and therefore one revolution per second is equal to 2π radians per second, or, in general,

$$\omega = 2\pi n \tag{28}$$

in which ω is the angular velocity of a body in radians per second and n is the angular velocity in revolutions per second.

Angular acceleration. — In many machines a part may rotate at a variable angular velocity. This is most strikingly illustrated by the motion of the balance wheel of a watch. The rate of change of the angular velocity of a body is called its *angular acceleration.* Thus an engine is started, and after six seconds the

fly wheel has an angular velocity of 4 revolutions per second ($= 25.13$ radians per second), so that the average angular acceleration of the wheel during the six seconds is 4.1888 radians per second per second. Of course, the fly wheel may have gained most of its angular velocity during a portion of the six seconds, so that 4.1888 radians per second per second is merely its average angular acceleration. The angular acceleration of a rotating body at a given instant is equal to the quotient $\Delta\omega/\Delta t$ where $\Delta\omega$ is the angular velocity gained during the short interval of time Δt.

62. Unbalanced torque and angular acceleration. Definition of moment of inertia. — When a wheel is set in rotation, an unbalanced torque must act upon the wheel. This is exemplified in the operation of spinning a top. When a rotating wheel is left to itself it loses its angular velocity and comes to rest on account of the friction of the wheel against the air and on account of the

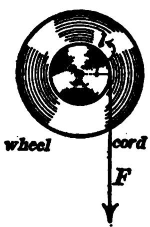


Fig. 53*a*.

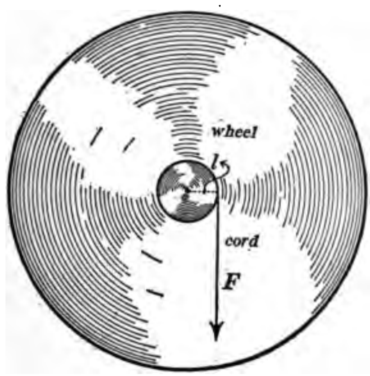


Fig. 53*b*.

friction of the shaft in its bearings. To maintain a steady motion of rotation of a wheel, a driving torque must act upon the wheel sufficient to balance the opposing torque due to friction.

The effect of an unbalanced torque is to change the angular velocity of a wheel, or, in other words, to produce angular acceleration, positive or negative as the case may be. The

angular acceleration of a given wheel is proportional to the unbalanced torque which acts upon the wheel, and a given unbalanced torque produces a small angular acceleration of a large heavy wheel, or a large angular acceleration of a small light wheel. Thus, if a cord be wrapped around the shaft upon which a wheel is mounted, a pull on the cord produces torque equal to Fl, Figs. 53*a* and 53*b*; and this torque imparts angular velocity to the small light wheel, Fig. 53*a*, at a rapid rate, whereas it imparts angular velocity to the large heavy wheel, Fig. 53*b*, at a much slower rate.

When a wheel is rotating every particle of the wheel moves at a definite linear velocity, and when the angular velocity of the wheel increases it is evident that the linear velocity of every particle of the wheel must increase; that is to say, *angular acceleration* of a wheel involves *linear acceleration* of every particle in the wheel, and it is possible to show the exact relation between angular acceleration and the unbalanced torque which produces it, by considering the linear acceleration of each particle in a wheel. The following discussion of this matter is the foundation of the dynamics of rotatory motion and it leads to a definition of what is called the *moment of inertia* of a wheel.

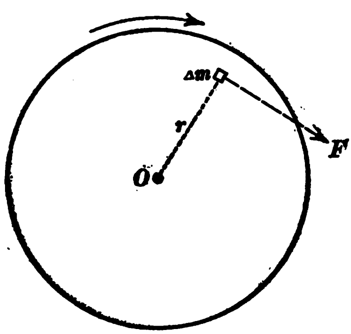


Fig. 54.

Figure 54 represents a wheel rotating n revolutions per second, or $2\pi n$ radians per second, about the axis O. The particle Δm describes a circular path of which the circumference is $2\pi r$, the particle traces this circumference n times per second, and therefore the linear velocity v of the particle is $2\pi rn$ centimeters per second, r being expressed in centimeters; but $2\pi n$ is equal to the angular velocity ω of the wheel in radians per second, and, therefore

$$v = r\omega \tag{29}$$

If the angular velocity of the wheel is changing, the linear velocity v of the particle m must change r times as fast as ω, inasmuch as v is always r times as large as ω. Therefore, representing the angular acceleration of the wheel by α (rate of change of ω) and representing the linear acceleration of the particle by a * (rate of change of v) we have

$$a = r\alpha \tag{30}$$

In order to produce the acceleration a of the particle, an unbalanced force F, see Fig. 54, must act on the particle in the direction of a, this force, expressed in dynamic units, must be equal to $\Delta m.a$ according to equation (3), and the torque action of this force is equal to $Fr(= \Delta m.a \times r)$; but a is equal to $r\alpha$ according to equation (30), so that $Fr = \Delta m.r^2\alpha$, or representing Fr by ΔT, we have

$$\Delta T = \alpha r^2 \cdot \Delta m$$

in which ΔT is that part of the unbalanced torque T acting on the wheel, which causes the linear acceleration of the given particle Δm. Consider in this way all of the particles of the wheel and we have

$$\Delta T = \alpha r^2 \cdot \Delta m$$

$$\Delta T_1 = \alpha r_1^2 \cdot \Delta m_1$$

$$\Delta T_2 = \alpha r_2^2 \cdot \Delta m_2$$

$$\Delta T_3 = \alpha r_3^2 \cdot \Delta m_3$$

etc., etc., whence, by adding, we have

$$T = \alpha(r^2 \cdot \Delta m + r_1^2 \cdot \Delta m_1 + r_2^2 \cdot \Delta m_2 + \cdots)$$

or

$$T = \alpha \Sigma r^2 \cdot \Delta m$$

or writing

$$K = \Sigma r^2 \cdot \Delta m \tag{31}$$

we have

$$T = K\alpha \tag{32}$$

* We are not concerned here with the *radial acceleration* of the particle Δm, since the radial acceleration is produced by unbalanced *radial forces* which have no torque action about O. Radial accelerations of the particles of a wheel have nothing to do with the angular acceleration of the wheel.

The quantity K, which is obtained by multiplying the mass of each particle of the wheel by the square of its distance from the axis and adding all of these products together, is called the *moment of inertia* of the wheel, and equation (32) shows that the unbalanced torque acting on a wheel is equal to the product of the moment of inertia of the wheel and the angular acceleration of the wheel.

Units involved in equations (31) *and* (32). If c.g.s. units are used throughout, then moment of inertia is expressed in grams × centimeters squared (gr. cm.^{2}), torque is expressed in dynes × centimeters and, of course, angular acceleration is expressed in radians per second per second. Equations (31) and (32) hold good, however, when moment of inertia is expressed in pounds × feet squared (lb. ft.^{2}), torque in poundals × feet and angular acceleration in radians per second per second. If torque is expressed in pounds-weight × feet, moment of inertia in pound-feet^2 and angular acceleration in radians per second per second, then equation (32) becomes

$$T = \frac{1}{32} K\alpha \tag{33}$$

approximately.

Example of the calculation of moment of inertia. The moment of inertia of a homogeneous solid of regular form can be calculated by the methods of calculus. Consider, for example, a long slim rod of length L and mass M rotating about its middle point O as shown in Fig. 55. The mass of the short portion dr is $M/L \times dr$ and its distance from O is r. Therefore, writing $M/L \times dr$ for Δm in equation (31) we have

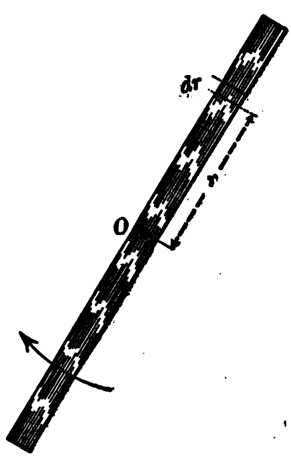


Fig. 55.

$$K = \frac{M}{L} \Sigma r^2 dr$$

but the sum (integral) $\Sigma r^2 dr$ between the limits $r = +L/2$ and $r = -L/2$ is equal to $\frac{1}{12}L^3$, so that $K = \frac{1}{12}ML^2$. The moments of inertia given in the table on following page were calculated in this way.

Radius of gyration. — The radius of gyration of a rotating body is the distance ρ from the axis of rotation at which the entire mass M

TABLE.

MOMENTS OF INERTIA OF SOME REGULAR HOMOGENEOUS SOLIDS.

Axis of rotation passing through center of mass.	Value of K.
Sphere of radius R and mass M	$\frac{2}{5}MR^2$
Cylinder of radius R and mass M, axis of cylinder is the axis rotation	$\frac{1}{2}MR^2$
Slim rod of length L and mass M, axis of rotation at right angles to rod	$\frac{1}{12}ML^2$
Rectangular parallelopiped of length L and breadth B, axis of rotation at right angles to L and B	$\frac{1}{12}M(L^2 + B^2)$

of the body might be concentrated without altering the moment of inertia of the body. If the entire mass M were concentrated at distance ρ from the axis, the moment of inertia would be equal to $M\rho^2$, according to equation (31). That is

$$K = M\rho^2 \tag{34}$$

or

$$\rho = \sqrt{\frac{K}{M}}$$

Using the values of K in the above table, this equation shows that the radius of gyration of a sphere is $\sqrt{\frac{2}{5}}$ times the radius of the sphere, the radius of gyration of a cylinder rotating about its axis of figure is $\sqrt{\frac{1}{2}}$ times the radius of the cylinder, and the radius of gyration of a long, slim rod rotating about an axis at right angles to the rod and passing through its center of mass is $\sqrt{\frac{1}{12}}$ times the length of the rod.

If a rotating body be imagined to be divided into particles of equal mass then the radius of gyration may be defined as the square-root-of-the-average-square of the distances of all the particles from the axis.

63. Kinetic energy of a rotating body. — A rotating wheel evidently stores kinetic energy because it can do work while being brought to rest. The kinetic energy of a rotating body is given by the equation

$$W = \tfrac{1}{2}K\omega^2 \tag{35}$$

in which everything is expressed in c.g.s. units. The proof of

this equation gives, perhaps, a clearer idea of the significance of moment of inertia than the discussion of equation (32) given in the foregoing article. Consider the rotating wheel shown in Fig. 54. The linear velocity of the particle Δm is $r\omega$, according to equation (29), and therefore the kinetic energy of this particle is

$$\Delta W = \tfrac{1}{2}\Delta m \cdot r^2\omega^2$$

according to equation (26).

Consider in this way all of the particles of the wheel and we have

$$\begin{aligned}
\Delta W &= \tfrac{1}{2}\Delta m \cdot r^2\omega^2 \\
\Delta W_1 &= \tfrac{1}{2}\Delta m_1 \cdot r_1^{\,2}\omega^2 \\
\Delta W_2 &= \tfrac{1}{2}\Delta m_2 \cdot r_2^{\,2}\omega^2 \\
\Delta W_3 &= \tfrac{1}{2}\Delta m_3 \cdot r_3^{\,2}\omega^2 \\
\text{etc.} &\quad \text{etc.}
\end{aligned}$$

whence, by adding, we have

$$W = \tfrac{1}{2}\omega^2\Sigma r^2 \cdot \Delta m$$

and by comparing this equation with equation (31) we have equation (35).

64. Relation between moments of inertia about parallel axes. — Let K be the moment of inertia of a body of mass M about a given axis passing through the center of mass of the body, and let K' be the moment of inertia of the body about another axis parallel to the first and distant d from the center of mass, then

$$K' = K + d^2M \tag{36}$$

Let O, Fig. 56, be the center of mass of the body, chosen as the origin of coordinates, let K be the moment of inertia of the body about an axis through O perpendicular to the plane of the paper, and let K' be the moment of inertia of the body about an axis through O' also perpendicular to the plane of the paper. Consider a sample particle of the body Δm, distant r from O and distant r' from O', and of which the coordinates are x and y.

By trigonometry we have

$$r'^2 = r^2 + d^2 - 2rd\cos\theta \tag{i}$$

From equation (31) we have

$$K' = \Sigma r'^2 \Delta m \tag{ii}$$

whence, substituting the value of r' from equation (i) we have

$$K' = \Sigma r^2 \cdot \Delta m + \Sigma d^2 \Delta m - 2d\Sigma r \cos\theta \cdot \Delta m \tag{iii}$$

but $\Sigma r^2 \cdot \Delta m$ is equal to K, and $\Sigma d^2 \Delta m$ is equal to $d^2 M$. Furthermore $\Sigma r \cos\theta \cdot \Delta m$

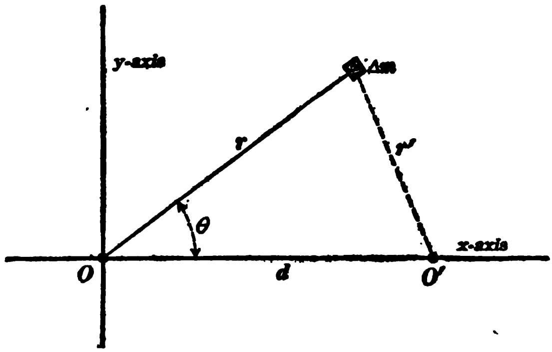


Fig. 56.

is equal to $\Sigma x \cdot \Delta m$, which is equal to zero according to equation (22) since the origin of coordinates is chosen at the center of mass of the body. Therefore equation (iii) reduces to equation (36)

65. Equivalent mass of a rolling wheel. — To set a rolling wheel in motion it is necessary not only to impart linear velocity to the wheel but also to set the wheel rotating, so that a given unbalanced force produces less acceleration than it would produce if the wheel were lifted from its track and allowed to move without rotation. Insofar as the relation between force (unbalanced) and acceleration is concerned, a rolling wheel may be treated as a body performing simple translatory motion by assigning to the wheel a mass in excess of its actual mass. This fictitious mass of a rolling wheel is called its *equivalent mass*, and it may be defined as that mass M which would store the same amount of kinetic energy as the rolling wheel at a velocity equal to the linear velocity v of the wheel. Thus we may write

$$\tfrac{1}{2}Mv^2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}K\omega^2 \tag{i}$$

in which m is the actual mass of the wheel, K is the moment of inertia of the wheel, and ω is the angular velocity of the wheel.

The angular velocity ω, however, satisfies the equation

$$v = r\omega \tag{29 bis}$$

and therefore, substituting v/r for ω in equation (i) we have

$$\tfrac{1}{2}Mv^2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\frac{K}{r^2}v^2$$

or

$$M = m + \frac{K}{r^2} \tag{37}$$

Examples. — (*a*) The rolling motion of the wheels of a railway train causes the train to behave, insofar as acceleration and kinetic energy relations are concerned, as if its mass were greater by the amount nK/r^2 than its actual mass, where n is the number of wheels, r is the diameter of the rolling circle of each wheel, and K is the moment of inertia of each wheel. In the case of an electric car with a geared motor, the moment of inertia of the motor

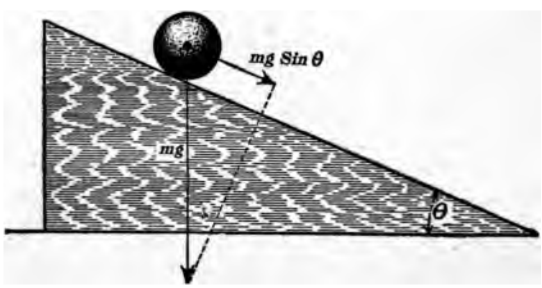


Fig. 57.

armature can be reduced to an equivalent moment of inertia of wheel and thus be included in the value of K in equation (37), by multiplying the moment of inertia of the motor armature by the square of the gear ratio (ratio of the diameters of the rolling circles of the two gears).

(*b*) Consider a metal sphere of mass m and radius r rolling down an inclined plane as shown in Fig. 57. The vertical pull of the earth mg has a component parallel to the plane which is equal to $mg \sin \theta$, and this force would cause the ball to move

with an acceleration equal to $g \sin \theta$ if it were not for the rotatory motion; but on account of the rotatory motion the sphere behaves as if its mass were $\frac{7}{5}$ times m, according to equation (37), and therefore it rolls down the plane with an acceleration of only $\frac{5}{7}$ of $g \sin \theta$. Friction is, of course, neglected.

A wheel and axle rolling on a track as shown in Fig. 58, has a

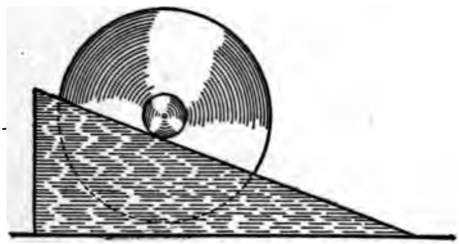


Fig. 58.

rolling circle of small radius r, so that its equivalent mass is very large, according to equation (37), and, therefore, such a wheel and axle rolls down an inclined plane with a very small acceleration.

66. Correspondence between translatory motion and rotatory motion. To every equation in translatory motion there is a corresponding equation in rotatory motion in which moment of inertia K takes the place of mass m, angle takes the place of distance, angular velocity ω takes the place of linear velocity v, angular acceleration α takes the place of linear acceleration a, and so on. The following table exhibits the pairs of corresponding equations.

TABLE

Translatory motion		Rotatory motion	
$F = ma$	(3)	$T = K\alpha$	(32)
$W = Fd$	(24)	$W = T\phi$	(i)
$P = Fv$	(25)	$P = T\omega$	(ii)
$W = \frac{1}{2}mv^2$	(26)	$W = \frac{1}{2}K\omega^2$	(35)
$F = -kx$	(15)	$T = -\kappa\phi$	(38)
$k = 4\pi^2 n^2 m$	(16)	$\kappa = 4\pi^2 n^2 K$	(39)

Of these equations, those numbered (i), (ii), (38) and (39) have not been previously discussed; equation (i) refers to the work

W done by the torque T in turning a body through an angle ϕ, axis of torque and axis of motion being coincident; and equation (ii) refers to the power P developed by a torque T which acts on a body rotating at angular velocity ω, axis of torque and axis of motion being coincident.

Equations (38) and (39) refer to harmonic rotatory motion, that is, to oscillatory motion about an axis, such as is exemplified by the motion of the balance wheel of a watch.

The equations of circular translatory motion correspond to the equations of the gyroscope to a limited extent as explained in Art. 72.

67. Rotatory harmonic motion. — Consider a weight suspended by a steel wire. The weight will stand in equilibrium with the wire untwisted. If the weight is turned around the wire as an axis through the angle ϕ from this equilibrium position, then the twisted wire will exert a torque T on the weight tending to turn it back, and this torque will be proportional to ϕ, that is

$$T = -\kappa\phi \tag{38}$$

in which the factor κ is a constant for a given wire; it is called the *constant of torsion* of the wire.

By analogy with harmonic translatory motion as discussed in Art. 42, it is evident from equation (38) that the suspended weight, if started, will oscillate about the wire as an axis, and that the number n of complete oscillations per second will satisfy the equation

$$\kappa = 4\pi^2 n^2 K \tag{39}$$

or, using $1/\tau$ for n, where τ is the period of one oscillation, we have

$$\kappa = \frac{4\pi^2 K}{\tau^2} \tag{40}$$

A weight hung by a wire and set oscillating about the wire as an axis, is called a *torsion pendulum.*

68. Use of the torsion pendulum for the comparison of moments of inertia. — The constant of torsion, κ, equations (39) and (40),

is nearly independent of the amount of weight supported by the wire, unless the weight becomes excessive, therefore if two bodies are hung from the same wire, one at a time, and their respective periods of torsional vibration τ and τ' observed, then from equation (40) we have

$$\kappa = \frac{4\pi^2 K}{\tau^2} \tag{i}$$

and

$$\kappa = \frac{4\pi^2 K'}{\tau'^2} \tag{ii}$$

whence

$$\frac{K}{K'} = \frac{\tau^2}{\tau'^2} \tag{iii}$$

from which K' may be calculated if K is known. For example, one of the suspended bodies may be a homogeneous circular disk of which the moment of inertia is known (see table in Art. 62).

69. The gravity of pendulum consists of a rigid body AB, Fig. 59, suspended so as to be free to turn about a horizontal axis O. Let C, Fig. 59, be the center of mass of the body. This point C is vertically below O when the body is in equilibrium. Let the body be swung to one side through the angle ϕ, as shown. Then the force, Mg, with which the earth pulls the body tends to swing the body *back* to the vertical position with a torque T which is equal to the product of Mg and the length of the arm aC'. But the distance aC' is equal to $x \sin \phi$, where x is the distance OC. Therefore

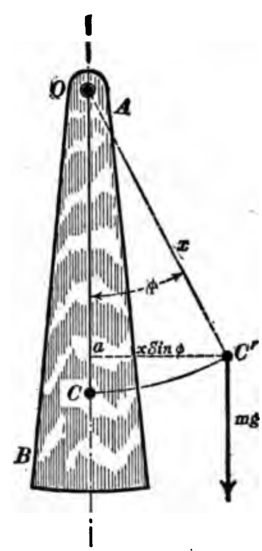


Fig. 59.

$$T = Mgx \sin \phi \tag{i}$$

When ϕ is small, then $\sin\phi = \phi$, very nearly, and equation (i) becomes

$$T = Mgx \cdot \phi \,^* \tag{ii}$$

Comparing this with equations (38) and (40), we find that

$$\frac{4\pi^2 K}{\tau^2} = Mgx \tag{41}$$

in which K is the moment of inertia of the body about the axis O, τ is the period of one complete pendulous vibration of the the body, and g is the acceleration of gravity.

A pendulum such as here described is sometimes called a *physical pendulum* to avoid confusion with the ideal *simple pendulum* described in Art. 43.

The simple pendulum. An ideal pendulum consisting of a particle of mass M suspended by a weightless cord, or rod, of length l is called a *simple pendulum.* The moment of inertia of such a pendulum about the supporting axis O is $K = Ml^2$, according to equation (31). Furthermore, the center of mass of a simple pendulum is, of course, at the center of the suspended particle. Therefore, for the simple pendulum, we may write Ml^2 for K, and l for x in equation (41), whence we have

$$\frac{4\pi^2 l}{\tau^2} = g \tag{42}$$

or

$$l = \frac{\tau^2 g}{4\pi^2} \tag{43}$$

in which l is the length of a simple pendulum, τ is the period of one complete vibration of the pendulum and g is the acceleration of gravity.

Equivalent length of a physical pendulum. The length of a simple pendulum which would have the same period of vibration as a given physical pendulum is called the *equivalent length* of the given physical pendulum. Now, according to equation (43),

* Of course this equation should be written $T = -Mgx \cdot \phi$ because T tends to reduce ϕ.

the length of a simple pendulum, of which the period of one vibration would be, τ is $l = \tau^2 g/4\pi^2$. Therefore, solving equation (41) for $\tau^2 g/4\pi^2$ ($= l$) we have

$$l = \frac{K}{Mx} \tag{44}$$

in which l is the equivalent length of a given physical pendulum, K is the moment of inertia of the pendulum about its axis of support, M is the mass of the pendulum, and x is the distance from the point of support to the center of the mass of the pendulum.

The point in the line OC, Fig. 59, which is at a distance $l(= K/Mx)$ from O is called the *center of oscillation* of the pendulum. This point is also called the *center of percussion* of the pendulum for the reason that if the pendulum is started or stopped by a horizontal hammer blow at this point no side force is produced on the supporting axis. See Art. 71.

70. The determination of gravity. — The most accurate determination of the acceleration of gravity is made by means of the pendulum. This determination would be a very simple thing if it were feasible to construct a simple pendulum, in which case equation (42) could be used for calculating gravity from the measured length, l, of the simple pendulum and its observed period τ. The determination of the acceleration of gravity by means of an actual pendulum depends, however, upon the determination of the moment of inertia of the pendulum, as is evident from equation (41), and the moment of inertia of a body cannot be determined with great accuracy. This difficulty is obviated by means of the so-called *reversion pendulum* which was devised by Henry Kater in 1818.

Fig. 60.

A simple form of Kater's pendulum is shown in Fig. 60. A stiff metal bar has two knife-edges, from either of which it may be swung as a pendulum, and two weights, WW, which may be adjusted until the period τ of one vibration of the pendu-

lum is the same whether it be swung from a or b. Then the distance between the knife-edges a and b is the equivalent length of the pendulum and may be used for l in equation (43).

Comparison of the values of gravity at two places by means of the pendulum. — If the same pendulum be swung at two places in succession and its respective periods τ and τ' observed, we have from equation (41)

$$\frac{4\pi^2 K}{\tau^2} = Mgx \tag{i}$$

and

$$\frac{4\pi^2 K}{\tau'^2} = Mg'x \tag{ii}$$

in which g and g' are the respective values of the acceleration of gravity at the two places. Dividing equation (i) by equation (ii), member by member, we have

$$\frac{g}{g'} = \frac{\tau'^2}{\tau^2}$$

From this equation the value of g may be accurately determined at any place in terms of its known value at another place, by observing the values of τ and τ' of an ordinary pendulum, every precaution being taken to avoid variations of dimensions of the pendulum due to temperature or to careless handling. Most of the gravity determinations of the United States Coast and Geodetic Survey are made in this way, the value of g at Washington having been once for all determined with the greatest possible accuracy by means of Kater's pendulum.

The opposite table gives the value of g in centimeters per second per second at several places as determined by the pendulum.

Theory of the reversion pendulum. — Consider a body of mass M, its center of mass at O, Fig. 61. Let O', O, and O'' be co-linear points; let τ' and τ'' be the vibration periods of the body swung as a pendulum from O' and O'' respectively; and let K, K', and K'' be the moments of inertia of the body about O, O', and O'' respectively. From equation (41) we have

$$\frac{4\pi^2 K'}{\tau'^2} = Mgx \tag{i}$$

TABLE.

Locality.	Latitude.	Longitude.	Elevation.	Value of g (not Reduced to Sea-level).
Boston, Mass.	42° 21′ 33″	71° 03′ 50″	22 meters.	980.382
Philadelphia, Pa.	39 57 06	75 11 40	16 "	980.182
Washington, D. C.	38 53 20	77 01 32	10 "	980.100
Ithaca, N. Y.	42 27 04	76 29 00	247 "	980.286
Cleveland, O.	41 30 22	81 36 38	210 "	980.227
Cincinnati, O.	39 08 20	84 25 20	245 "	979.990
Terre Haute, Ind.	39 28 42	87 23 49	151 "	980.058
Chicago, Ill.	41 47 25	87 36 03	182 "	980.264
St. Louis, Mo.	38 38 03	90 12 13	154 "	979.987
Kansas City, Mo.	39 05 50	94 35 21	278 "	979.976
Denver, Col.	39 40 36	104 56 55	1638 "	979.595
San Francisco, Cal.	37 47 00	122 26 00	114 "	979.951
Greenwich	51 29 00	0 00 00	47 "	981.170
Paris	48 50 11	2 20 15	72 "	980.960
Berlin	52 30 16	13 23 44	35 "	981.240
Vienna	48 12 35	16 22 55	150 "	980.852
Rome	41 53 53	12 28 45	53 "	980.312
Hammerfest	70 40 00	22 38 00	——	982.580

and

$$\frac{4\pi^2K''}{\tau''^2} = Mgy \tag{ii}$$

From equation (36) we have

$$K' = K + x^2M \tag{iii}$$

$$K'' = K + y^2M \tag{iv}$$

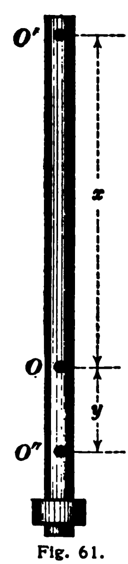


Fig. 61.

Substituting these values of K' and K'' in (i) and (ii), we have

$$\frac{4\pi^2(K + x^2M)}{\tau'^2} = Mgx \tag{v}$$

$$\frac{4\pi^2(K + y^2M)}{\tau''^2} = Mgy \tag{vi}$$

Eliminating K/M from (v) and (vi), we have

$$\frac{4\pi^2(x^2 - y^2)}{x\tau'^2 - y\tau''^2} = g \tag{45}$$

If $\tau' = \tau''$, we may cancel $(x - y)$, *if* $(x - y)$ *is not equal to zero*, giving

$$\frac{4\pi^2(x + y)}{\tau'^2} = g \tag{46}$$

(1) If the pendulum has been adjusted by repeated trial, so that $\tau' = \tau''$, then

equation (46) enables the calculation of g, when $(x+y)$ and τ' have been observed.

(2) If the pendulum has not been adjusted, equation (45) enables the calculation of g, when x, y, τ', and τ'' have been observed.

(3) If the pendulum has been roughly adjusted, so that τ' and τ'' are nearly equal, then *equal and opposite errors in x and y very nearly annul each other in their influence upon the value of g as calculated by equation* (45). Therefore equation (45) gives g very accurately when τ' and τ'' are nearly equal, $(x+y)$ being measured with great accuracy, and x measured roughly. The value of y is taken from $(x+y)-x$, so that its error may counteract the error due to the roughly measured value of x. The position of the center of mass O, Fig. 61, is found with sufficient accuracy for the rough measurement of x by balancing the pendulum horizontally on a knife edge.

Note.—When $x=y$, equation (45) becomes indeterminate, giving $0/0=g$, and in this case equation (46) is not necessarily true, since it has been derived from equation (45) by cancelling $(x-y)$, which is zero.

71. Motion of a rigid body when struck with a hammer.— When an unbalanced force continues to act upon a body for an appreciable length of time, the problem of determining the motion of the body is complicated by the fact that, as the body moves, the force generally changes its point of application, or its value, or its direction, or all three of these things may change simultaneously. The force due to a hammer blow, however, is of such short duration that the actual movement of the body during the time that the force acts is negligible and the problem of finding the motion produced by the hammer blow is quite simple. A hammer blow is called an *impulse** and it is measured by the product of the average value, F, of the force exerted by the hammer and the short time t that the force continues to act; that is, an impulse is expressed in terms of force multiplied by time, in *dyne-seconds*, if c.g.s. units are employed.

A rigid stick, AB, Fig. 62*a*, is struck with a hammer in the direction of the arrow, h, at a point distant x above the center of mass, O, of the stick. The motion imparted to the stick by the blow is a combination of translatory motion and rotatory motion, but the combination of a constant translatory motion and a constant rotatory motion is exactly the kind of motion

* The impulse of a hammer blow, when the hammer is brought to rest by the blow, is equal to the momentum of the hammer. See Art. 47.

which is performed by a rolling wheel, and therefore the hammer blow causes the stick to move as if the stick were attached to a circular hoop, *CC*, and this hoop allowed to roll on a straight rail. The center of the rolling circle, *CC*, is at the center of mass of the stick, and the radius, y, of the rolling circle depends upon the distance, x, and upon the ratio of the moment of inertia of the stick (about O) to the mass of the stick, according to equation (47). At the instant of the hammer blow the motion of the stick is equivalent to a simple motion of rotation about the point O''.

To analyze the effect of the hammer blow, the translatory mo-

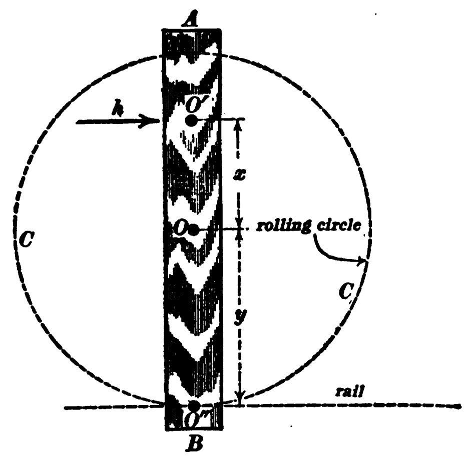


Fig. 62*a*.

tion and the rotatory motion may be treated separately. Regarding the translatory motion, we know, from Art. 48, that the velocity imparted to the center of mass is the same as if the whole mass of the body were concentrated there and acted upon directly by the total force of the hammer. Let F be the average force due to the hammer, and t the time (very short) that it continues to act. Then F/M is the acceleration of the center of mass, and F/M multiplied by t is the velocity imparted to the center of mass.

As to the rotatory motion of the body, it is evident that Fx is

the torque about O due to the force of the hammer, so that Fx/K is the angular acceleration of the body during the time t, and Fx/K multiplied by t is the angular velocity imparted to the body by the hammer blow.

Now the whole body is moving to the *right* at a velocity Ft/M on account of the translatory motion, and any point at a distance r below O is moving to the *left* at a linear velocity equal to r times the angular velocity, Ftx/K, therefore, for the point O'', which is for the moment stationary, we must have, writing y for r,

$$\frac{Ftxy}{K}=\frac{Ft}{M}$$

or

$$xy=\frac{K}{M} \tag{47}$$

which determines the radius y of the rolling circle when x and K/M are given.

The problem of the base-ball bat. — At the instant that a base-ball bat strikes a ball, the motion of the bat is a simple motion of rotation about a certain point O'' Fig. 62*b*; and, if the distances x and y satisfy equation (47), then the effect of the impact of bat and ball is to reduce the angular velocity of the bat about the point O'' without moving the point O''. The point of a bat which must strike a ball so that the impact may have no tendency to change the position of the point about which the bat is rotating at the instant of impact, is called the *center of percussion* of the bat. The position of the center of percussion depends of course upon the position of the point O'' about which the bat is rotating at the instant of impact.

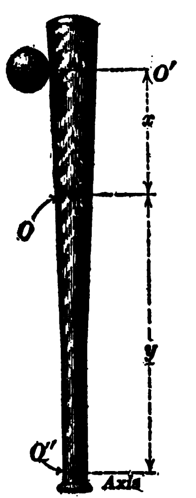


Fig. 62*b*.

72. Precessional rotatory motion. The foregoing articles refer to rotation about a fixed axis, or, as in the case of a rolling wheel, to rotation about an axis which performs translatory motion. The axis of a rotating body may, however, change its direction continuously. We shall discuss here only the comparatively simple case * in which a symmetrical body spins about its axis of symmetry while at the same time the axis of spin rotates uniformly. This rotation of the axis of spin is called *precession*, and the axis about which the axis of spin rotates is called the *axis of precession.*

The *gyroscope* consists of a heavy wheel mounted on an axle which is pivoted in a metal supporting ring, as shown in Fig. 63. The wheel is set in rapid rotation by wrapping a cord on the axle and giving the cord a vigorous pull. When the wheel is thus set rotating, the direction of the axle remains unaltered as long as no external twisting force, or torque, acts upon it; *an unbalanced torque is necessary to change the* **direction** *of the axis of a rotating body, just as an unbalanced force is required to change the* **direction** *of translatory motion of a particle.*

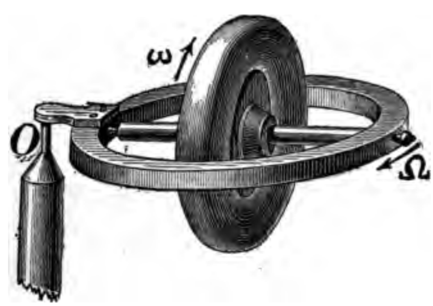


Fig. 63.

In order to describe precisely how an unbalanced torque changes the direction of the axis of a rotating body, it is very convenient to represent *angular velocity* and *torque* by lines in a diagram. To represent an angular velocity by a line, draw the line in the direction of the axis of spin and of such length as to represent to scale the value of the angular velocity in radians per second; to represent a torque by a line, draw the line in the

* The student is referred to Poinsot's *Theorie Nouvelle de la Rotation des Corps*, which is perhaps the most intelligible account of the motion of a non-symmetrical rigid body.

direction of the axis of the torque and of such length as to represent to scale the value of the torque in dyne-centimeters. In each case an arrow-head is to be placed on that end of the line towards which a right-handed screw would travel if turned in the direction of rotation in the one case, or if turned in the direction of the torque in the other case.

Figure 64*a* is a top view of the gyroscope; the metal ring rests upon a supporting pivot underneath the ring at *O*, and the line *OP*, Fig. 64*b*, represents the angular velocity ω of the spinning

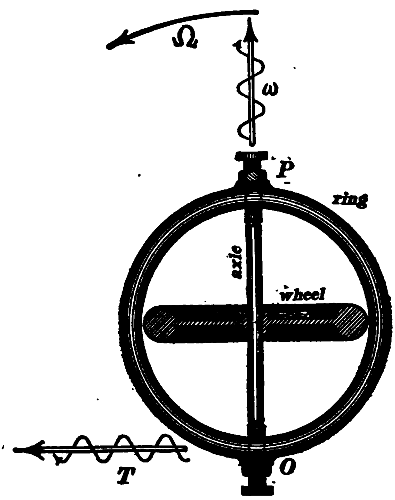


Fig. 64*a*.

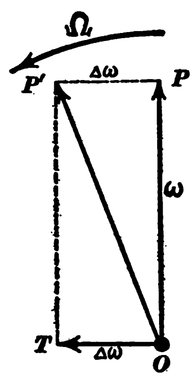


Fig. 64*b*.

wheel at a given instant. The pull of the earth on the wheel and ring produces an unbalanced torque about the axis *OT*, Fig. 64*a*. The effect of this unbalanced torque, during a short interval of time, is to impart to the wheel an additional angular velocity $\Delta\omega$ about the axis *OT*, and the resultant angular velocity is then about the axis *OP'*, Fig. 64*b*; that is, *the effect of the unbalanced torque T is to cause the axis of spin to sweep around O in the direction of the arrow* Ω.

This effect of an unbalanced torque upon a rapidly rotating body is also exemplified by the motion of a spinning top. Thus

the line OP, Fig. 65, represents the angular velocity of a spinning top. The vertical pull of the earth, mg, produces an unbalanced torque about O, and the angular velocity produced by this unbalanced torque, by being added continuously to OP as a vector, causes the axis of spin OP to sweep round the vertical axis OV in the direction indicated by the arrow Ω.

The force required to constrain a particle to a circular orbit depends upon the mass of the particle and upon the linear acceleration which is involved in the continual change of direction of the velocity of the particle. The torque required to produce pre-

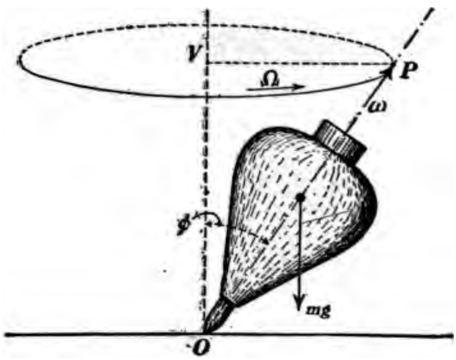


Fig. 65.

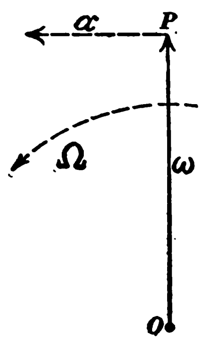


Fig. 66.

cession of a spinning body depends upon the moment of inertia of the body and upon the angular acceleration which is involved in the continual change of direction of the axis of spin. Precessional motion of a spinning body corresponds to circular translatory motion.

The torque required to produce precessional rotatory motion may be derived* from equation (32) as follows: Let the line

* This derivation is correct only when the rotating body is symmetrical with respect to the axis of spin and also to the axis of precession, as in Figs. 64 and 69; and it is approximately correct in a case like Fig. 65, provided the angular velocity of spin is very much greater than the angular velocity of precession. A precessional motion of the top, Fig. 65, about OV tends to increase the angle ϕ independently of the torque due to the force mg, and therefore the precessional motion of OP is more rapid than would be produced by mg alone if the top were symmetrical about OV.

OP, Fig. 66, represent the angular velocity of spin, ω, of a wheel, and let Ω be the angular velocity at which the axis *OP* sweeps round *O*. *The line OP represents* ω, *and the linear velocity of the end P of the line represents the angular acceleration* α *which is involved in the continual change of direction of the axis OP.* Therefore, according to the principles enunciated in Art. 21, we have,

$$\alpha = \omega\Omega \tag{i}$$

when the axis of spin is at right angles to the axis of precession, as in Fig. 64; and

$$\alpha = \omega\Omega \sin\phi \tag{ii}$$

when the axis of spin makes an angle ϕ with the axis of precession as in Fig. 65. Using these values of α in equation (32) we have

$$T = \omega\Omega K \tag{iii}$$

when the axis of spin is at right angles to the axis of precession; and

$$T = \omega\Omega K \sin\phi \tag{iv}$$

when the axis of spin makes an angle ϕ with the axis of precession.

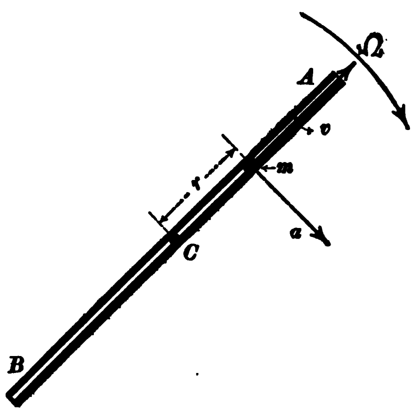


Fig. 67.

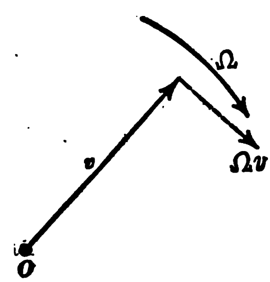


Fig. 68.

The above analysis of the action of the gyroscope will hardly be convincing to the beginner on account of the fact that the action is analyzed in terms of the rather

complicated and unfamiliar ideas, angular velocity, angular acceleration, moment of inertia and torque; it is, therefore, desirable to analyze the action of the gyroscope in terms of the fundamental ideas of linear velocity and acceleration, mass, and linear force. The analysis of the action of the gyroscope in terms of linear velocity and acceleration depends upon a relation which is sometimes called Coriolis' law. Given a straight tube AB, Fig. 67, which is rotating about the axis C at angular velocity Ω as indicated in the figure. In this tube is a ball m which is moving away from C at velocity v (if the ball were moving towards C its velocity v would be considered as negative). Under these assumed conditions *the sidewise acceleration, a, of the ball m is equal to* $2\Omega v$, that is

$$a = 2\Omega v \tag{v}$$

To derive this relation, the sidewise acceleration a may be considered in two parts. In the first place we have the acceleration which is associated with the continual change of direction of the radial velocity v of the ball. This accleration is equal to Ωv as shown in Fig. 68, and as explained in Arts. 21 and 38. In the second place, as the ball gets farther and farther away from the axis C, its actual sidewise velocity due to the rotation of the tube increases, but this sidewise velocity is equal to Ωr,

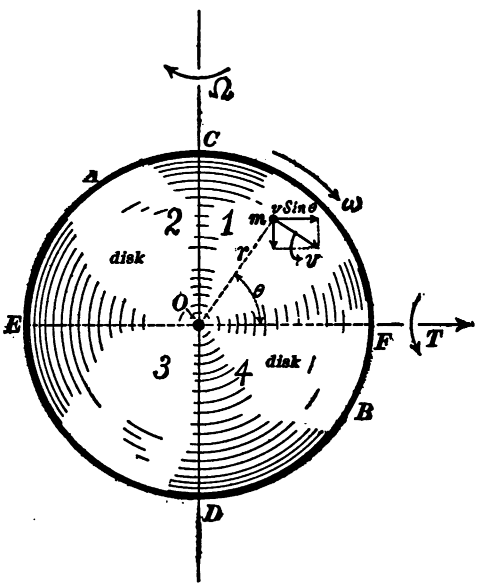


Fig. 69.

according to equation (29), and therefore, since v is the rate at which r is changing, it is evident that Ωv is the rate at which the sidewise velocity, Ωr is changing, as explained in Art. 20. The relation $a = 2\Omega v$ is used in the discussion of the motion of steam engine governors, where the governor balls have a motion of rotation combined with a motion towards or away from the axis.

Consider now a circular disk AB, Fig. 69, spinning at angular velocity ω about its axis of figure O, and let the axis O be turning about CD at angular velocity Ω. Consider a sample particle m of the disk at a distance r from O as shown in Fig. 69. The velocity of m is $r\omega$, and the component of this velocity which is away from the axis CD is $r\omega \sin\theta$; and, therefore, the precessional rotation about the line CD involves an acceleration of m towards the reader which, according to equation (v), is

$$a = 2\omega\Omega r \sin\theta \tag{vi}$$

It may be easily seen that this acceleration is *towards the reader* in quadrants 1 and 2, and *away from the reader* in quadrants 3 and 4, and, therefore, the forces required to produce these accelerations constitute a torque about the axis EF as indicated by the arrow T.

73. Examples of precessional rotation. (*a*) *The precession of the earth's axis.* — The attraction of the sun keeps the earth in its orbit. The force of attraction of the sun upon the bulging

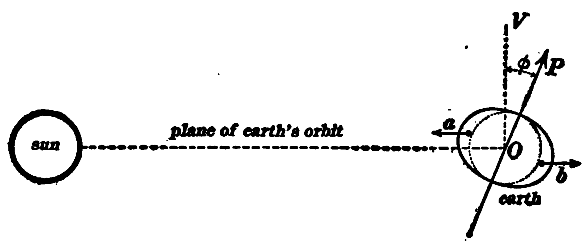


Fig. 70.

equatorial portion a, Fig. 70, is greater than the "centrifugal force" due to orbital motion of the earth, and the force of attraction on the bulging equatorial portion b is less than the "centrifugal force." Therefore, the earth is acted upon by an unbalanced torque about O which causes the earth's axis to describe a cone about the line OV which is at right angles to the plane of the earth's orbit. The action of the moon is here ignored for the sake of simplicity.

(*b*) *A coin rolling along the floor* is, of course, rotating, and the instant the coin begins to be inclined to either side, the unbalanced torque due to gravity causes a precessional movement of the axis of the coin, and the coin describes a curved path in consequence of this precession.

(*c*) *Rotating parts of machines on ship-board.* — The pitching and rolling of a vessel at sea causes, at each instant, a certain angular velocity Ω of the axis of a rotating machine part, and an unbalanced torque is immediately brought into existence. For example, when a steamer turns round, the propeller and propeller shaft change direction continuously, when a steamer rolls the axis of a dynamo armature which is athwart ship changes its direction periodically, when a steamer pitches the axis of the propeller and propeller shaft changes its direction periodically. In the case of a steamer driven by steam turbines the propeller shaft turns at high speed and the rotating member of the turbine is quite heavy, so that the pitching motion of such

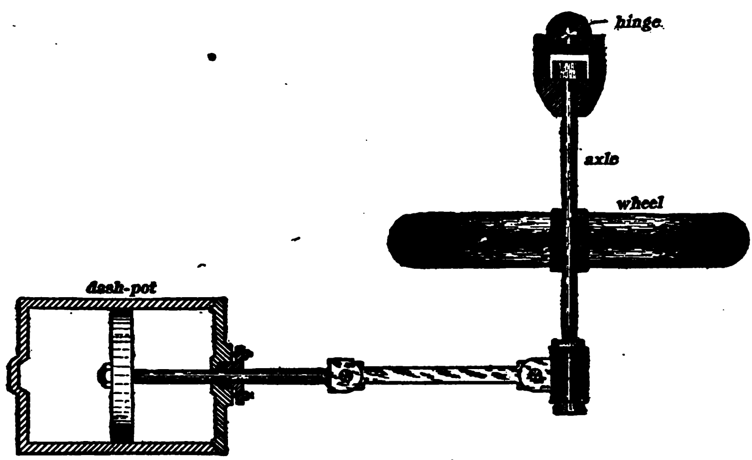


Fig. 71.

a vessel would produce excessively large forces at the bearings which support the shaft. A turbine torpedo-boat of the British Navy went down in a heavy sea in 1899 or 1900, being probably broken in two by the very great forces produced by the pitching of the boat, and the consequent angular motion of the propeller shaft, forces which, perhaps, were not duly considered in the designing of the hull and supporting structure of the shaft.

(*d*) *When a locomotive turns a curve* the wheels and axles turn

about a vertical axis, and a torque is brought into existence. When a side-wheel steamboat turns, the precession of the heavy side-wheels and shaft causes the boat to list.

(*e*) *The use of the gyroscope (or gyrostat as it is sometimes called), for preventing the rolling of a ship at sea.* A rapidly rotating wheel is hung from a hinge so that its axis may swing back and forth in a vertical plane, including the keel of the boat, as shown in Fig. 71. The lower end of the axis is attached to a rod which connects to a piston in what is called a dash-pot. When the vessel rolls about the keel as an axis, the axis of the gyrostat oscillates back and forth, and the effect of the friction of the piston in the dash-pot is the same as if the rolling of the ship were hindered by excessive friction, and thereby the motion of rolling is greatly reduced. A small German torpedo boat, 115 feet long by 12 feet beam, was recently equipped with a gyrostat arranged as shown in Fig. 71.* The gyrostat wheel was 3.3 feet in diameter, it had a mass of 1,100 pounds and it was driven at a speed of 1,600 revolutions per minute. The effect of this arrangement is shown in Fig. 72, in which abscissas measured from the line marked 0° represent angular amplitudes of rolling oscillations. Above the point *A* the curve shows the rolling when the gyrostat is inoperative,

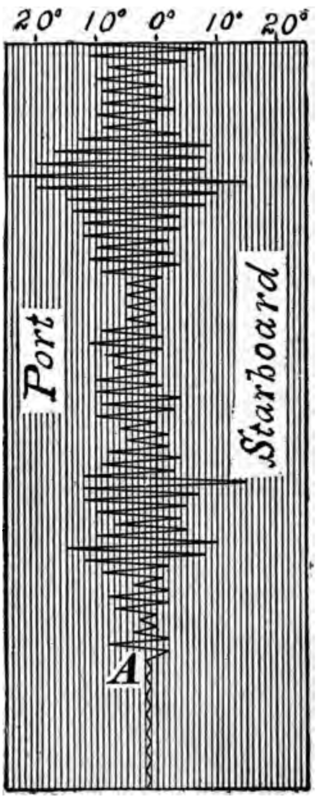


Fig. 72.

* See paper by Otto Schlick, translated in *Scientific American Supplement*, for January 26, 1907.

and below the point A the curve shows the rolling when the gyrostat is in action.

KINEMATICS OF A RIGID BODY.*

74. Motion of a rigid body in a plane. — A rigid body is said to move in a plane when all points of the body which lie in the plane remain in it. For example, a rotating wheel moves in a plane, the connecting rod of a steam engine moves in a

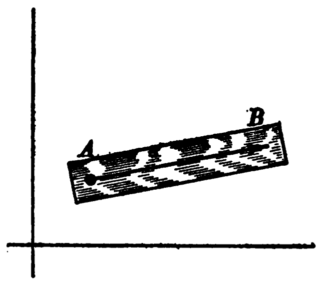


Fig. 73.

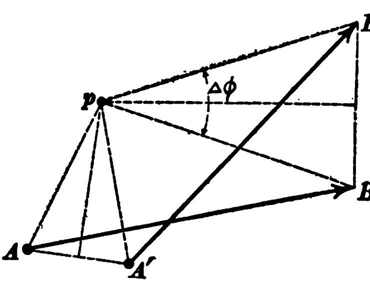


Fig. 74.

plane. Consider a rigid body AB, Fig. 73, moving in the plane of the paper. The position of the body is completely indicated by the position of the line AB fixed in the body. This line is called the *index line.*

After any change in the position of a rigid body moving in a plane, a certain line in the body, perpendicular to the plane, is in its initial position, and the given displacement is equivalent to a rotation about that line as an axis. Let AB and $A'B'$, Fig. 74, be the positions of the index line before and after the displacement. Join AA' and BB'. Erect perpendiculars from the middle points of AA' and BB' intersecting at p. From the similarity of the triangles pAB and $pA'B'$ it is evident that the same part of the body is at p before and after the displacement, and that the line through p perpendicular to the paper is the line about which the body may, by simple rotation, move from its initial to its final position. The angle $\Delta\phi$ of this rotation is the angle subtended by AA' or BB' as seen from p.

75. The instantaneous motion of a rigid body moving in a plane in any manner, is a motion of rotation about a definite line called the *instantaneous axis* of the motion. Let the displacement, shown in Fig. 74, be that which takes place in a short interval of time Δt; then $\Delta\phi / \Delta t$ is the instantaneous angular velocity of the body, and the line through p, perpendicular to the paper, is the instantaneous axis. During

* The discussion of the dynamics of a rigid body should properly be preceded by a discussion of thc kinematics of a rigid body. This, however, has not been done because most of the discussion of the dynamics of a rigid body can be based upon the simple idea of rotation about a fixed axis. Thus the rotatory motion of a rolling wheel is in no way different from what it would be if the translatory motion did not exist.

a finite interval of time the motion of a body may be irregular, but the motion of a body during an interval of time approaches uniformity as that interval approaches zero. Therefore the motion of a body during a short interval of time is the simplest motion which can produce the actual displacement which occurs during the interval.

76. Composition of angular and linear displacements. Consider an angular displacement $\Delta\phi$ of a body about the point p, Fig. 75, bringing the point O to O'; and the linear displacement Δl parallel and equal to $O'O$, bringing O' back to O. These two displacements are together equivalent to an angular displacement $\Delta\phi$ about O, bringing Op to Op'. Let the distance of p from the line OO' be r; then, if $\Delta\phi$ is small, $\Delta l = r\Delta\phi$.

77. Resolution of motion in a plane.— From Arts. 75 and 76 it follows that the instantaneous motion of a rigid body in a plane may be resolved into a motion of rotation about an arbitrary point combined with a certain linear velocity. Consider the actual displacement represented in Fig. 75, namely, a rotation about O bringing p to p'. This displacement is equivalent to an equal angular displacement $\Delta\phi$, about the

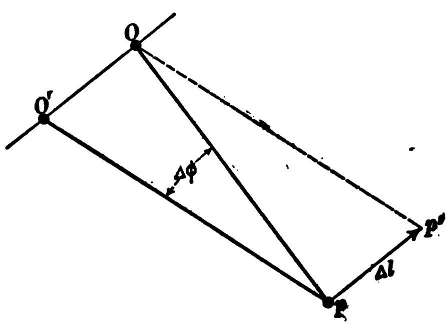


Fig. 75.

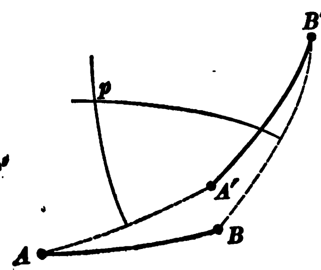


Fig. 76.

arbitrary point p, together with the linear displacement $O'O$ or pp'. Let this linear displacement be Δx, and let Δt be the interval which elapses during the displacement. The actual angular velocity $\Delta\phi/\Delta t$ about the point O (the instantaneous center) is equivalent to an angular velocity $\Delta\phi/\Delta t$ about the point p combined with a linear velocity $\Delta x/\Delta t$ parallel to pp'.

78. Motion of a rigid body with one point fixed. — If a rigid body, one point of which is fixed, is displaced in any manner whatever, a certain line in the body will be in its initial position after the displacement, and the given displacement will be equivalent to a rotary movement about this line as an axis.

Proof. — Consider a spherical shell of the body having its center at a fixed point. Let AB, Fig. 76, be an arc of a great circle on this spherical shell; the position of AB fixes the position of the body, and AB is called the *index line*. Let the movement of the body bring AB to $A'B'$. Connect AA' and BB' by arcs of great circles. Draw great circles bisecting AA' and BB' at right angles. The point p at the intersection of these circles bisecting AA' and BB' has the same position relative to AB and $A'B'$, so that this point of the shell is in its initial position, and

the line drawn from the center of the spherical shell to the point p is the axis about which the given movement can be produced by rotation.

The instantaneous motion of a rigid body about a fixed point is a motion of simple rotation at definite angular velocity about a definite line called the instantaneous axis of the motion.

79. Vector addition of angular velocities. — Consider an angular velocity about the axis a, Fig. 77, and another angular velocity about the axis b; the two angular

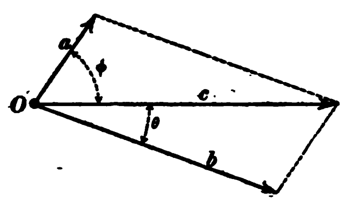


Fig. 77.

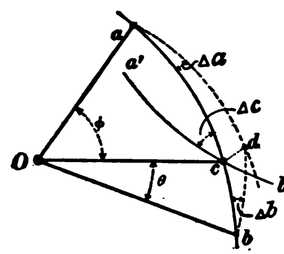


Fig. 78.

velocities are together equivalent to an angular velocity about the axis c, the respective angular velocities being proportional to the lengths of the lines a, b and c.

Outline of proof. — Imagine a sphere constructed with its center at O, Fig. 77, and let a and b, Fig. 78, be the points where the lines a and b, Fig. 77, cut the sphere. Imagine a very small rotation Δa about Oa followed by a very small rotation Δb about Ob, bringing the great circle ab to the position $a'b'$. The

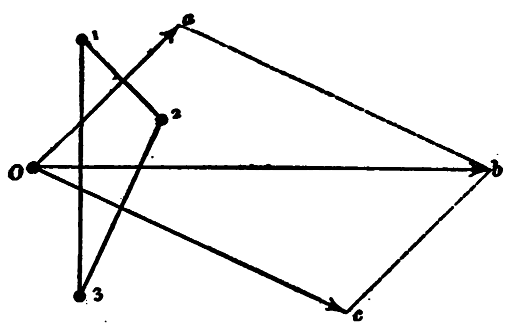


Fig. 79.

point of intersection of ab and $a'b'$ is the point where the resultant axis Oc cuts the sphere, and the angle Δc is the amount of rotation about Oc which is equivalent to the two rotations Δa and Δb; then it can be shown that the three angles Δa, Δb and Δc are related to each other as the lengths of the three lines a, b and c, Fig. 77, and that the two arcs ac and cb are equal to ϕ and θ respectively of Fig. 77.

80. Vector addition of torques. — Let the lines a and b, Fig. 79, represent two given torques and let it be required to show that a and b are together equivalent to

the torque c. Draw the lines 1–2, 2–3, and 1–3 perpendicular to and bisected by a, b and c respectively. The lengths of these lines are proportional to the lengths a, b and c. Imagine the torque a to be due to a unit upward force at 1 and a unit *downward* force at 2 (upward and downward being perpendicular to the plane of the paper), then the torque b is equivalent to a unitt of *upward* force at 2 and a unit of downward force at 3; but the upward force and downward force at 2 annul each other, so that we have left only a unit of upward force at 1 and a unit of downward force at 3, which give a torque about the line c proportional to the length of c.

PROBLEMS.

86. A body starts from rest and after 10 seconds it is rotating 55 revolutions per second. What is the average angular acceleration? Express the result in radians per second per second.

87. In what terms is moment of inertia expressed: (a) When length is expressed in centimeters and mass in grams? (b) When length is expressed in inches and mass in pounds? (c) When length is expressed in feet and mass in pounds? The unit moment of inertia in case (a) is the c. g. s. unit. How many c. g. s. units of moment of inertia are there in the unit involving the inch and the pound, and in the unit involving the foot and the pound?

88. Calculate the moment of inertia of a uniform slim rod, length 3.1 feet ($= l$) and mass 3.6 pounds ($= m$), about an axis passing through the center of the rod and at right angles to the length of the rod.

(a) Calculate K from the formula $K = ml^2/12$.

(b) Calculate K approximately by multiplying the mass of each 0.1 foot of the rod by the square of its estimated mean distance from the center of the rod.

(c) Calculate the radius of gyration of the rod.

89. Calculate the moment of inertia of a circular disk, radius 1.7 feet, mass 4.25 pounks, about the axis of figure.

(a) Calculate K from the formula given in the table in Art. 62. The circular disk is of course a very short cylinder.

(b) Calculate the radius of gyration of the disk.

90. (a) Calculate the moment of inertia of the rod, problem 88, about an axis passing through the end of the rod and perpendicular to the rod.

(*b*) Calculate the moment of inertia of the disk, problem 89, about an axis passing through the edge of the disk parallel to the axis of figure of the disk.

91. A circular disk, 5 feet diameter, weighing 1,200 pounds is mounted upon a shaft 6 inches in diameter. The disk, set rotating at 500 revolutions per minute and left to itself, comes to rest in 75 seconds. Calculate average (negative) angular acceleration while stopping, calculate average torque acting to stop the disk, and calculate the frictional force at the circumference of the shaft.

92. What is the kinetic energy of the disk specified in problem 91 when the speed is 500 revolutions per minute?

93. A metal disk 12 inches in diameter and weighing 25 pounds, has a cylindrical hub projecting on each side. Each hub is 1 inch in diameter and weighs ¼ of a pound (total mass 25.5 pounds). What is the moment of inertia of the whole?

The hubs of this disk roll on a track which drops 1 inch vertically in each foot of horizontal distance, find how fast the disk gains linear velocity in rolling down this track.

94. A slim rod 2 feet long and weighing 2.5 pounds is suspended by a wire. The wire is attached to the middle of the rod and the rod hangs in a horizontal position. The rod, set vibrating about the wire as an axis, makes 50 complete vibrations in 10 minutes, 25 seconds. What torque would be required to twist the wire through one complete turn?

95. An irregular body is suspended by the same wire that is specified in problem 94, and, set vibrating about the wire as an axis, it makes 37 complete vibrations in 10 minutes. What is its moment of inertia?

96. A uniform slim rod 4 feet long is hung as a gravity pendulum at a point distant 6 inches from the end of the bar. Calculate its equivalent length as a pendulum.

97. A pendulum clock rated in Boston and carefully transported to Hammerfest would gain how many seconds per day? *See table in Art. 70.*

98. The connecting rod of a steam engine weighs 50 pounds, its center of mass is distant 18 inches from the center of the hole

which fits the crank pin, and when it is swung as a gravity pendulum about the point *a*, Fig. 98*p*, it makes 90 complete vibra-

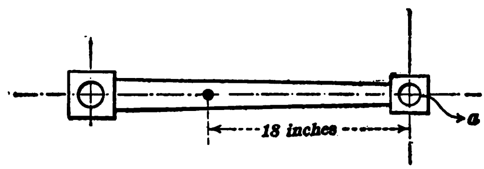


Fig. 98*p*.

tions in one minute. The diameter of the hole is 1¾ inches. What is the moment of inertia of the connecting rod about its center of mass?

99. A stick four feet long and weighing 10 pounds is held vertically and struck a horizontal blow with a hammer at a point 18 inches from the upper end, which is released at the instant of the blow. The impulse of the blow is 80 pounds-weight-seconds. Find the translatory velocity imparted to the stick, find the angular velocity about its center of mass, and find the position of the point in the stick which remains stationary for a moment after the hammer blow.

100. Find the distance from the axis of suspension, of the slim rod described in problem 96, to the point where the rod may be struck horizontally with a hammer without causing a side force to be exerted on the axis of suspension. Compare this distance with the "equivalent length" of the rod as a gravity pendulum.

101. A water wheel is connected to its belt pulley by a shaft. Find the torque, in pound-feet and in pound-inches, tending to twist the shaft when the water wheel develops 200 horse-power at a speed of 600 revolutions per minute.

Note. — The two equations $W = T\phi$ and $P = T\omega$ correspond exactly to equations (24) and (25) as explained in Art. 66. The only difficulty involved in the use of the equations $W = T\phi$ and $P = T\omega$ is to keep the units straight, as it were.

102. An electric motor, running at 900 revolutions per minute, develops 15 horse-power. Find the torque with which the field

magnet acts upon the rotating armature, neglecting friction. Express the result in pound-feet and in pound-inches.

103. The armature shaft of a ship's dynamo is athwart ship, and the armature is driven clockwise as seen from the port side of the vessel. Describe accurately the forces with which the bearings act upon the armature shaft as the vessel rolls. Specify the directions of these forces when the port side of the vessel is rising, and when the port side of the vessel is falling.

This problem refers to the forces which arise from the rotatory motion of the armature. The port side of a vessel is on the left hand of a person facing the bow.

104. A side-wheel steamboat is suddenly turned to port, and the gyrostatic action of the paddle wheels causes the boat to list. In which direction does the boat list, to starboard or port? Why?

105. The vessel described in problem 104 is steered in a circle 150 feet in radius at a velocity of 25 feet per second, and the vessel lists 5° because of the gyrostatic action of the paddle wheels and shaft. To produce a 5° list when the boat is standing still requires a weight of 10 tons to be shifted from the center of the boat to a point 15 feet from the center. The paddle wheels make 75 revolutions per minute. Find the moment of inertia of the axle and wheels.

106. A locomotive rounds a curve of radius 528 feet at a speed of 30 miles per hour. The diameter of the driving wheels is 6 feet and each pair of drivers and the connecting axle has a moment of inertia of 37000 pound-feet^2. Find the torque acting on each pair of drivers due to the precession. How does this precession modify the force with which the wheels push on the two rails?

107. A torpedo boat makes a complete turn in 84 seconds and its propeller rotates at a speed of 270 revolutions per minute. The moment of inertia of the propeller is 2000 pound-feet^2. Required the precessional torque on the propeller shaft. In what direction does this torque tend to bend the shaft?

108. A high speed engine with its shaft athwart ship, makes 240

revolutions per minute. The rim of the fly-wheel has a radius of 3 feet and a mass of 600 pounds. Calculate the moment of inertia of the wheel (rim). The maximum angular velocity attained by the vessel in rolling is $\frac{1}{10}$ radian per second. Calculate the maximum torque acting on the fly-wheel shaft due to gyrostatic action and specify the direction of the torque.

CHAPTER VIII.

ELASTICITY (STATICS).

81. Stress and strain. — When external forces act upon a body and tend to change its shape, the body is distorted more or less, and the external forces are balanced by the tendency of the distorted body to return to its original shape. The distortion of a body always brings forces into action between the contiguous parts of the body throughout. These force actions between contiguous parts of a distorted body are called *internal stresses*; and the total reaction of the distorted body, which balances the external distorting force, is called the *integral stress* of the body.

The actual movement of the point of application of an external force which distorts a body is called the *integral strain* of the body, and the change of shape of each small part of the distorted body is called the *internal strain*. Thus, the elongation of a wire under tension, the shortening of a column under compression, the angular movement of the end of a rod under torsion, the depression of the middle of a beam which is loaded at its center, and the decrease of volume of a body which is subjected to hydrostatic pressure are integral strains; and the total stretching force acting on the wire, the total load on the column, the total torque tending to twist the rod, the total load at the middle of the beam, and the hydrostatic pressure which acts on a body, are integral stresses. In each of these cases, furthermore, each small part of the body is distorted, and force actions exist between contiguous parts of the body throughout. These are called the internal strains and the internal stresses respectively.

82. Homogeneous and non-homogeneous stresses and strains. — It is generally the case in a distorted body, that each small part of the body is differently distorted, and that the internal stress varies from point to point in the body. For example, the dif-

ferent parts of a bent beam, or of a twisted rod, are differently distorted, and the internal stress varies from point to point; the pressure of the atmosphere decreases and the air becomes less and less dense with increasing altitude above the level of the sea; the pressure at a point in a body of water increases with the depth beneath the surface, and the water is more and more compressed as the pressure increases; the stress in a long cable, which is suspended in a mine shaft, increases from the lower end upwards, and the extent to which each portion of the cable is stretched increases with the stress.

When the force action between contiguous parts of a body is the same at every point in the body, the stress is said to be

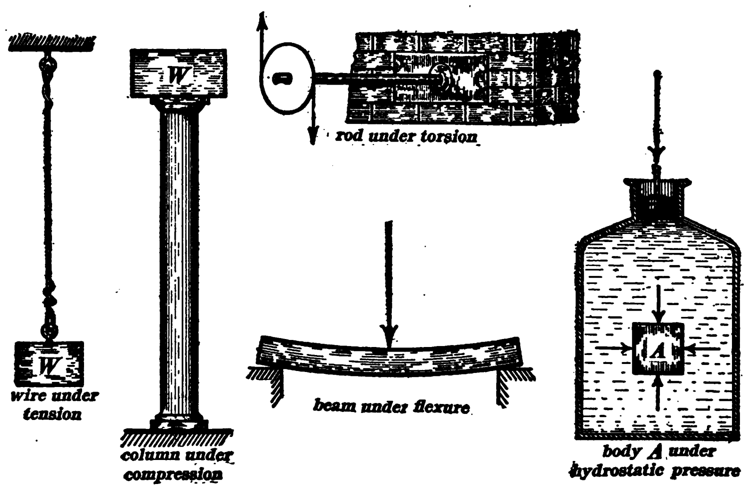


Fig. 80.

homogeneous; and when every part of a body is similarly distorted, the strain is said to be *homogeneous.* Thus each part of a rod under tension or compression is similarly distorted as shown in Figs. 86*a* and 86*b*, and the force action or stress is the same at every point as shown in Figs. 85*a* and 85*b*; the steam in a boiler is under the same pressure throughout (gravity negligible), and the degree of compression of every portion of the steam is the same; the water in the high pressure cylinder of a hydraulic

press is under the same pressure throughout (gravity negligible), and the degree of compression of every portion of the water is the same.

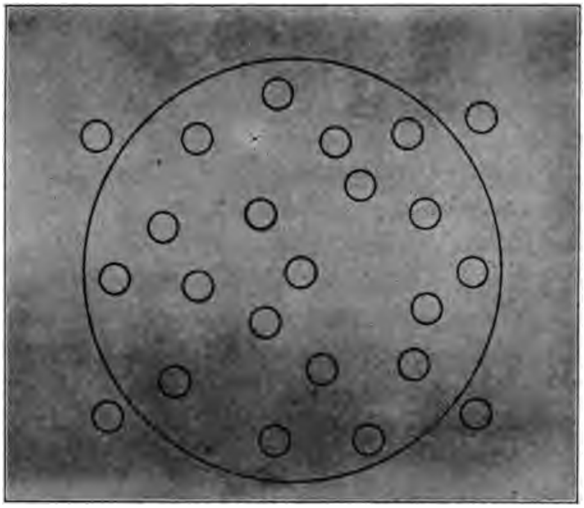


Fig. 81.

The distinction between homogeneous and non-homogeneous strains is shown in Figs. 81, 82, and 83, which are photographs

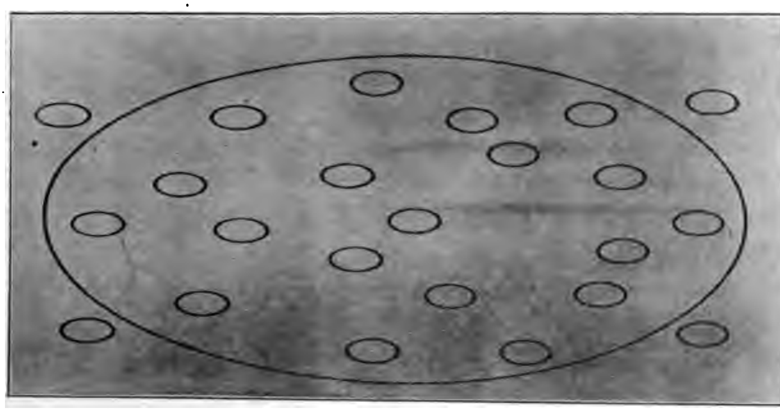


Fig. 82.

of a thin rubber sheet upon which a large circle and a number of small circles were drawn. Figure 81 shows the unstrained sheet,

Fig. 82 shows the sheet homogeneously strained, and Fig. 83 shows the sheet non-homogeneously strained. The small circles are changed to ellipses in Fig. 82 and in Fig. 83, but in Fig. 82 the ellipses are all alike and their axes are in the same direction, whereas, in Fig. 83, some of the ellipses are more elongated than others and their axes are not parallel. In a homogeneous strain a *large* portion of a substance is distorted in a manner exactly similar to the distortion of each *small* portion of the substance. This is shown in Fig. 82 in which the large ellipse is exactly the same shape as the small ones. In a non-homogeneous strain a *large* portion of a substance is irregularly distorted. This is

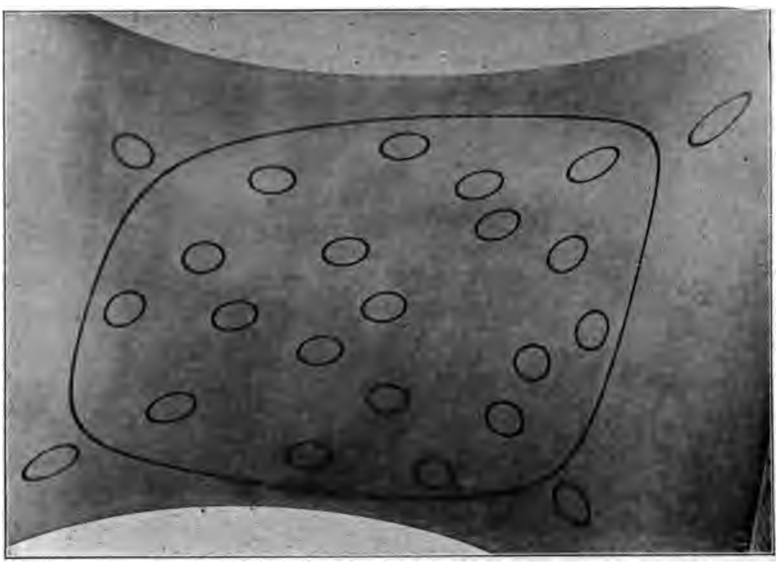


Fig. 83.

shown in Fig. 83 by the irregular curve into which the large circle has been converted by the strain. *It is a fact of fundamental importance in the theory of elasticity, that, however irregularly a body may be distorted, any small portion of the body suffers that simple kind of distortion which changes a sphere into an ellipsoid, or which, in the case of a thin sheet of rubber, changes a circle into an ellipse.* That is, the change of shape of any small portion of

a distorted body consists of an increase or decrease of linear dimensions in three mutually perpendicular directions, and, in some cases, this simple kind of distortion is accompanied by a slight rotation of the small parts of the body. Thus, in Fig. 90, which represents a portion of a bent beam, the short straight lines were all horizontal or vertical in the unbent beam.

The effect of a sharp groove in a body which is under stress is a matter of very great practical importance. The effect is, in general, to produce an excessive concentration of stress in the material at the bottom of the groove, and a crack or fracture is almost sure to develop, unless the material is plastic so that the bottom of the groove is broadened by yielding. Consider, for example, a beam in which a sharp groove is cut, as shown in Fig. 84. The fine lines in this figure represent the lines of stress

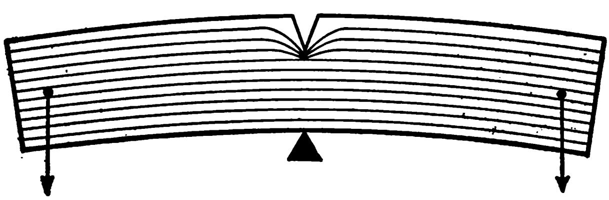


Fig. 84.

when the beam is bent, and the crowding together of these lines as they pass under the groove represents the concentration of stress above referred to.

The most striking illustration of the effect of a sharp groove in a body under stress is furnished by a piece of glass in which there is a minute crack. A piece of glass without a crack will stand a very considerable stress, but if the stress "flows" round the end of a crack, the stress is concentrated and the crack extends indefinitely. A pane of window glass or a glass tumbler is worthless when a crack once starts. A less familiar illustration of the effect of a sharp groove is furnished by the method commonly employed for breaking a bar of steel; thus, a steel rail, which normally withstands the tremendous stresses due to

the weight of a locomotive, can be broken in two by a hammer blow if a nick is made across the top of the rail with a sharp chisel. Sharp re-entrant angles are always carefully avoided in the designing of those parts of structures which are intended to sustain stress.

83. Solids and fluids. — Everyone is familiar, in a general way, with the three classes of substances, solids, liquids, and gases. A *solid* can withstand, for an indefinite length of time, a stress which tends to change its shape. A solid which recovers from distortion (strain) when stress ceases to act, is said to be *elastic*. Thus good spring steel recovers almost completely from a moderate amount of distortion when the distorting force (stress) ceases to act. A solid which does not recover from strain when the stress ceases to act, is said to be *plastic*. Thus lead and wax are plastic solids. No solid, perhaps, recovers completely from distortion; and, on the other hand, every plastic substance, perhaps, is slightly elastic. Thus the best spring steel does not completely recover from even a slight distortion, and when the distortion is great the steel takes a very decided permanent set; and even wax is slightly elastic, as is shown by the distinct metallic ring of a large cake of bees wax or paraffine when it is struck with a hammer.

A *fluid* is a substance which, at rest, cannot sustain a stress which tends to change its shape. While a fluid is actually changing shape, however, it does sustain a stress which tends to change its shape. Thus a stream of syrup falling from a vessel is under tension like a stretched rope, and the effect of this tension is to continually lengthen each portion of the stream of syrup as it falls. A fluid at rest always pushes normally against every portion of a surface which is exposed to the action of the fluid. Thus the steam in a boiler pushes outwards on the boiler shell at each point, the water in a vessel pushes normally against the walls of the vessel at each point, and the atmosphere pushes normally against every portion of an exposed surface. A fluid in motion, however, may not push normally against an exposed

surface. Thus, a water pipe is subject only to a bursting force, if the water is at rest; but if the water flows through the pipe, it has a slight tendency to drag the pipe along with it.*

A *liquid* is a fluid, like water or oil, which can have a free surface, such as the surface of water in a glass. A *gas*, on the other hand, is a fluid which completely fills any containing vessel.

84. Hooke's law. Elastic limit. — Robert Hooke discovered, in 1676, that what we have called the integral strain of a body is quite accurately proportional to what we have called the integral stress.† Thus, the elongation of a wire under tension is proportional to the stretching force, the shortening of a loaded column is proportional to the load, the angular movement of the end of a rod under torsion is proportional to the torque which acts on the rod, and the depression of a loaded beam is proportional to the load.

Elastic limit. — Hooke's law is quite accurately true for distinctly elastic substances like steel, but it does not apply to plastic substances, and even for elastic substances like steel there is a limit, called the *elastic limit*, beyond which stress and strain are no longer even approximately proportional. When an elastic substance is strained beyond its elastic limit it does not return to its original size or shape when the stress ceases to act, but takes what is called a *permanent set.* Liquids and gases, however, return to their exact initial volume when relieved from pressure, provided the temperature has not changed, that is, liquids and gases may be said to be perfectly elastic, but when a liquid or gas is compressed the diminution of volume (integral strain) is not proportional to the increase of pressure (integral stress) except when the increase of pressure is fairly small. This is at once evident in the case of gases when we consider that they conform to Boyle's law as explained in Art. 101.

* A jet of water issuing from the end of a pipe pushes backwards on the pipe, as every fireman knows. This backward force is due to the normal force with which the water pushes on the inner walls of the pipe where the pipe bends.

† It follows from this experimental fact that the strain at each point of a distorted elastic body is proportioual to the stress at that point.

85. Limitations and plan of this chapter.—The phenomena which are associated with the distortion of bodies are excessively complicated. Let one consider the swaying of objects in the wind, the bending and compression of structures under load and their vibration with sudden variations of load; let one think of all the familiar properties of brittle substances like chalk and glass, of plastic substances like clay and wax and of elastic substances like steel and rubber; let one consider that all of the phenomena of sound are due to the vibrations of bodies, and to wave movements in the air, and, in many cases, to wave movements in water and in solids, all of which have to do with distortion and compression; and let one think that local changes of shape and compression and expansion are involved inextricably in nearly every case of flow of air and water. Let one think of all of these things and then consider whether it is not necessary to bring the mind to some narrow view before any clear line of argument can be pursued relative thereto!

Of all the great variety of solid substances, having almost every imaginable degree of elasticity, plasticity, hardness, and brittleness, and ranging in strength from sun dried clay to the toughest steel, we are here concerned only with the behavior under stress of those which are used as materials of construction; and, in addition, we are here concerned with the tendency of increase of pressure to reduce the volumes of liquids and gases.

A substance, like wood, which has a grained structure, is said to be *aeolotropic* (pronounced ē′-o-lo-trop′-ic). Most crystalline substances, and rolled and drawn metal are aeolotropic. A substance, like glass or water, which does not have a grained structure, is said to be *isotropic*. The behavior under stress of aeolotropic substances is very complicated; these complications need not be considered, however, for practical purposes, because substances having a grained or fibrous structure are generally subjected to stresses parallel to the grain, as in beams, and ropes, and wires. The difference between aeolotropic substances and isotropic substances is ignored in this chapter.

Types of stress and strain. — In the discussion of the behavior of bodies under stress, it is necessary to consider three simple types of stress and strain. Thus we have *longitudinal stress* and *longitudinal strain*, which is the type of stress and strain in a rod under tension or in a column under compression; we have *hydrostatic pressure* and *isotropic* * *strain*, which is the type of stress and strain in a body subjected to hydrostatic pressure; and we have *shearing stress* and *shearing strain*, which is the type of stress and strain which exists (non-homogeneously) in a twisted rod. A discussion of the first two types of stress and strain is sufficient for most practical purposes, and, therefore, the discussion of shearing stress and shearing strain is given in small type preceeding the outline of the general theory of stress and strain.

Treatment of non-homogeneous stresses and strains. — In the following discussion, the behavior of a substance under each type of homogeneous stress and strain, is first considered, and the ideas so developed are used as a basis for the discussion of important cases of non-homogeneous stresses and strains. For example, the discussion of the bent beam is based upon the discussion of homogeneous longitudinal stress and strain, and the discussion of the twisted rod is based upon the discussion of homogeneous shearing stress and strain.

LONGITUDINAL STRESS AND STRAIN.

86. Longitudinal stress. — Figure 85a represents a portion of a rod under tension. Let F be the total force tending to stretch the rod, and let A be the sectional area of the rod; then the stretching force per unit of sectional area is F/A $(= P)$, and the force action between contiguous portions of the rod is as follows: Imagine a horizontal unit of area q anywhere in the material of the rod, the material on the two sides of q exerts a pull P across q, as shown in Fig. 85a; imagine a vertical unit of area q' drawn

* When an isotropic substance, such as glass, is subjected to a hydrostatic pressure the substance is reduced in volume without being changed in shape. Such a strain is called an isotropic strain, for want of a better name.

anywhere in the material of the rod, the material on the two sides of q' does not exert any force at all across q'. The force acting across any horizontal area of a units is, of course, equal to Pa. The force per unit area, P, is the *measure** of the longitudinal stress, and the direction of P is called the *axis of the stress.*

A rod under tension may be considered as a case of *positive* longitudinal stress, and a rod or column under compression may be considered as a case of *negative* longitudinal stress.

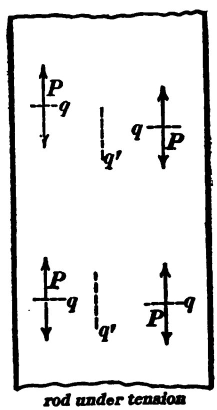


Fig. 85*a*.

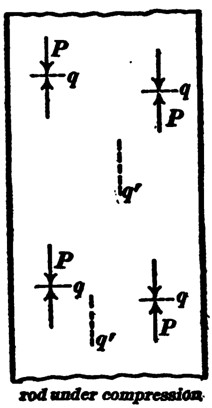


Fig. 85*b*.

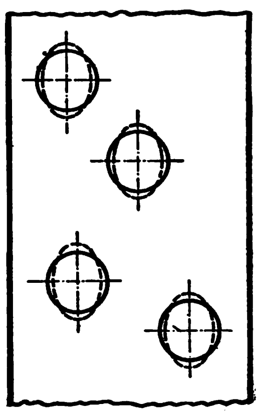
87. Longitudinal strain.—A rod under tension is longer than when it is not under tension, and, since each unit portion of the rod must be equally stretched, it is evident that the increase of length of the rod is proportional to its total initial length. Therefore it is most convenient to express the increase of length as a fraction of the total initial length; thus, a stretch of 2 thousandths means an increase of length equal to 2 thousandths of the total initial length of a rod. Let L be the initial length of a rod and let l be its increase of length under tension, then l/L $(=\beta)$ is the increase of length expressed as a fraction of the initial length, and, of course, the ratio of $L + l$ to L is equal to

* That is, the number which is used to specify the value of the stress. See Art. 11.

$1 + \beta$. The fraction $\beta\ (= l/L)$ is used as the *measure* of the longitudinal strain.

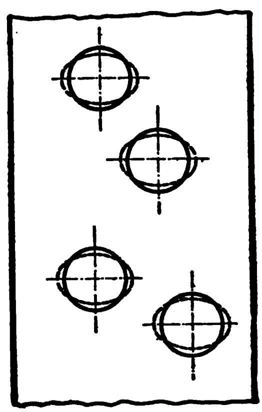
The character of the distortion of the parts of a rod under tension is shown in Fig. 86*a*, any spherical portion of the material of the rod becomes an ellipsoid of revolution. The major axis of the ellipsoid is $1 + \beta$ times the diameter of the original sphere.

Lateral contraction of a stretched rod. Poisson's ratio. — When a rod is stretched, it contracts laterally as indicated by the dotted



rod under tension

Fig. 86*a*.



rod under compression

Fig. 86*b*.

ellipses in Fig. 86*a*. This lateral contraction of a stretched body is very strikingly shown by a stretched rubber band. The lateral contraction d of a stretched rod is most conveniently expressed as a fraction of the original diameter D of the rod, and the fraction d/D we may represent by the letter β'. The elongation of a rod per unit length (β) bears for a given substance a fixed ratio to the lateral contraction per unit of diameter (β'), and this ratio is called *Poisson's ratio*; β is approximately four times as large as β' for steel, brass, and copper.

88. Stretch modulus of a substance. — The elongation of a rod under tension is proportional to the stretching force (Hooke's

law). Therefore the stretching force divided by the elongation is a constant for a given rod, and if we divide *stretching force per unit area, P, by elongation per unit of original length* $\beta\,(=l/L)$, the result is a constant which depends upon the material of the rod, but which is independent of the size and length of the rod. This constant, which is represented by the letter E, is called the *stretch modulus** of the material of which the rod is made. That is

$$E=\frac{P}{\beta} \tag{48}$$

The stretch modulus may be defined in a slightly different way as the factor which, multiplied by the elongation per unit length, gives the stretching force per unit sectional area of a rod under tension $(E\beta=P)$; and it is evident from this definition that E is expressed in units of force per unit of area, inasmuch as β is a ratio of two lengths (l/L); in fact E is the force per unit sectional area which *would* double the length of a rod if the elongation would continue to be proportional to the stretching force. This, of course, is not true for elongations of more than a few parts per thousand.

89. Determination of stretch modulus. — The stretch modulus may be determined by applying a known stretching force F to à rod of known length L and known sectional area A, and observing the increase of length l. Then $P=F/A$, and $\beta=l/L$, so that $E=P/\beta=FL/Al$. An easier method for determining

TABLE.

Values of the stretch modulus of various substances.

(In pounds-weight per square inch.)

Copper (drawn)	17,700,000
Steel (rolled)	29,800,000
Wrought iron	29,600,000
Cast iron	16,000,000
Glass	9,600,000
Oak wood	1,450,000
Poplar wood	750,000

* Often called Young's modulus, or "the modulus" of elasticity by engineers.

the stretch modulus of a substance is by observing the deflection of a loaded beam as explained in Art. 91.

90. Potential energy of longitudinal strain.—The potential energy per unit volume of a stretched (or compressed rod) is equal to one half the product of the stress P and the strain β, or it is equal to one half of the product of the stretch modulus E and the square of the strain β. That is

$$W = \tfrac{1}{2}P\beta \tag{49}$$

or

$$W = \tfrac{1}{2}E\beta^2 \tag{50}$$

in which W is the potential energy per unit of volume of a stretched (or compressed) rod, P is the stretching (or compressing) force per unit sectional area of the rod; β is the increase (or decrease) of length per unit of original length, and E is the stretch modulus of the material. If P and E are expressed in pounds-weight per square inch, W is expressed in inch-pounds of energy per cubic inch of material.

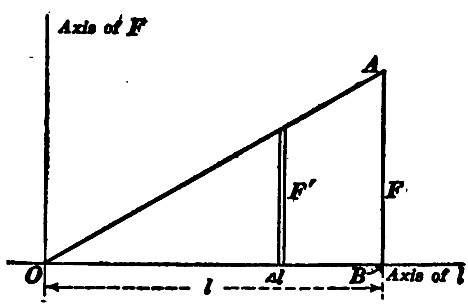


Fig. 87.

Proof.—The increase of length l of a rod is proportional to the stretching force F, so that by plotting corresponding values of l and F we have a straight line OA, as shown in Fig. 87.

Imagine the stretching force to increase slowly from zero to F, the stretch at the same time increasing from zero to l. Let F' be an intermediate value of F, and let Δl be a very small increase of l due to a slight increase of F', then $F' \cdot \Delta l$ is the work done

on the rod during the very slight increment of stretch Δl. But $F' \cdot \Delta l$ is the area of the narrow parallelogram shown in Fig. 87, and therefore the total work done on the rod while the stretching force increases from zero to F is the total area OAB Fig. 87, which is equal to $\frac{1}{2}Fl$, so that the work done per unit volume of the rod is $\frac{1}{2}Fl$ divided by the volume AL of the rod. That is

$$W = \frac{1}{2}\frac{Fl}{AL} = \frac{1}{2}P\beta.$$

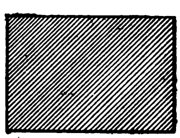
This proof may be stated in a slightly different way thus: The *average* value of the stretching force between zero stretch and the given stretch l, is $\frac{1}{2}F$, which, multiplied by the elongation l, gives the work done on the rod.

Equation (50) is derived from equation (49) by substituting $E\beta$ for P according to equation (48).

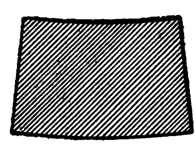
91. Discussion of a bent beam.* — The simplest case of a bent beam is that which is shown in Fig. 88 in which a long beam is laid horizontally across two supports SS, and bent by hanging weights on the projecting ends as shown. The bending moment or torque of each weight is equal to Wx, and this bending torque acts on every portion of the beam between the supports SS, so that this portion of the beam becomes an arc of a circle. All of the filaments in the upper part of the beam are elongated, all of

* When a beam is bent the stretched filaments on one side of the beam contract laterally and the compressed filaments on the other side of the beam expand laterally and the section of a beam originally square becomes distorted somewhat as shown in Fig. 89. This is very clearly shown by bending a rectangular bar of rubber, a lead

Fig. 89.



Section of beam before bending.



Section of beam after bending (exaggerated).

pencil eraser for example. This distortion of the section of a bent beam is usually very slight and it is neglected in the above discussion.

the filaments in the lower part of the beam are shortened, and certain filaments *pp*, which lie in what is called the *median* line or surface of the beam, remain unchanged in length. The beam

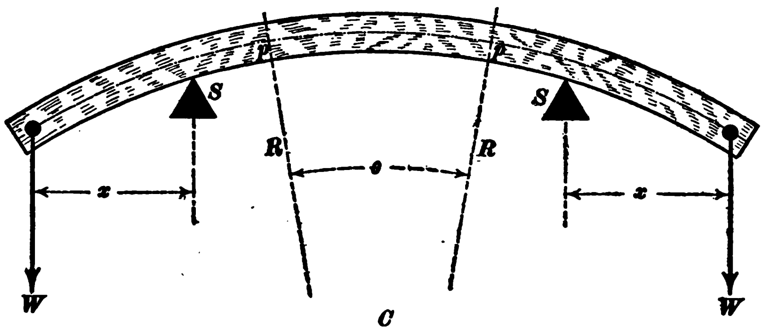


Fig. 88.

is everywhere under longitudinal strain as shown in Fig. 90; and the force action between contiguous parts of the beam is as shown in Fig. 91. These figures may be understood by comparing them

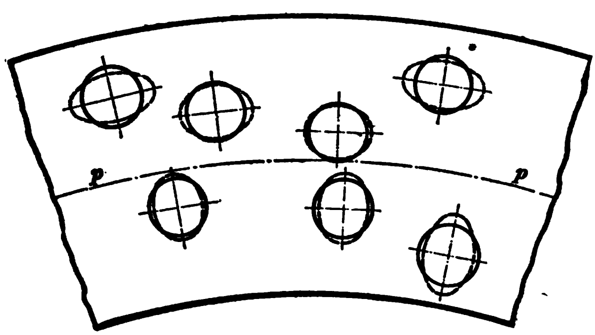


Fig. 90.

with Figs. 85 and 86. To find the value of the stress and strain at each point in the bent beam proceed as follows:

Consider the portion of the beam which lies between the radii R and R Fig. 88. This portion is shown to a larger scale in Fig. 92. Let R be the radius of curvature of the median line *pp*, then the length of *pp* is equal to $R\theta$ *which is the original length of*

every filament of the beam between the radii R and R Fig. 88. Consider a filament of the beam at a distance y above the median line, y being considered negative for filaments below the median line. The radius of curvature of this filament is $R + y$ and its length is

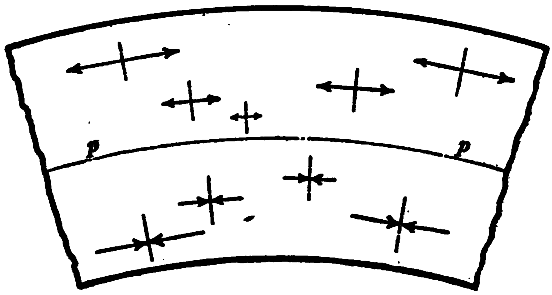


Fig. 91

$(R + y)\theta$. Therefore, the increase of length of this filament due to the bending of the beam is $y\theta$, and, expressing this increase of length as a fraction of the original length $R\theta$, we have

$$\beta = \frac{y}{R} \tag{i}$$

which expresses the value of the longitudinal strain at any point

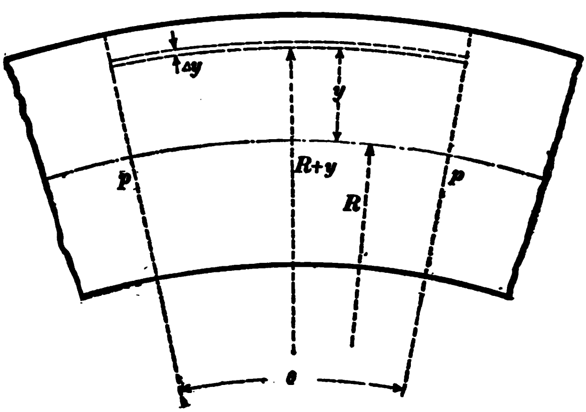


Fig. 92.

in the bent beam, β being positive where y is positive and negative where y is negative.

The longitudinal stress P (force per unit of area, as shown in Fig. 91) at each point of the beam is equal to $E\beta$, according to equation (48), where E is the stretch modulus of the material of the beam. Therefore, using y/R for β, we have

$$P = \frac{E}{R} \cdot y \qquad \text{(ii)}$$

The total force action across a complete section ab of the beam shown in Fig. 93, that is, the total force action of the por-

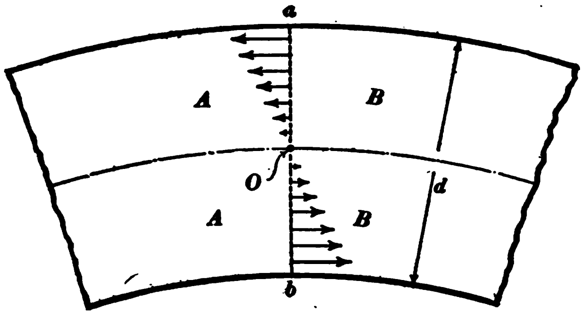


Fig. 93.

tion AA upon the portion BB, is a torque about an axis O perpendicular to the plane of the figure. This axis is shown as the line OO in Fig. 94, which is a sectional view of the beam,

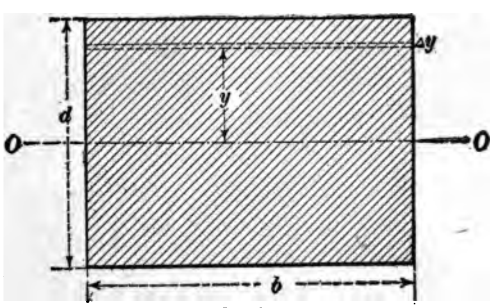


Fig. 94.

b being the breadth of the beam and d its depth. The torque action is expressed by the equation

$$T = \tfrac{1}{12} \frac{bd^3E}{R} \qquad \text{(iii)}$$

But the torque action across every section of the beam between the two supports SS in Fig. 88 is equal to Wx, so that

$$Wx = \tfrac{1}{12} \frac{bd^3E}{R} \qquad \text{(iv)}$$

from which E may be calculated when W, x, b, d and R are known.

Derivation of equation (iii). — The portion of the beam between the two parallel lines Δy, in Fig. 94, has a sectional area equal to $b \cdot \Delta y$, and the force action across each unit of area of this portion of the beam is Ey/R, according to equation (ii) above. Therefore the total force action across the area $b \cdot \Delta y$ is $Eby \cdot \Delta y / R$, which, multiplied by the lever arm y, gives the torque action about OO which is due to the portion of the beam $b \cdot \Delta y$. Therefore,

$$\Delta T = \frac{Eb}{R} \cdot y^2 \cdot \Delta y$$

whence by integrating between the limits $y = -d/2$ and $y = +d/2$, we have equation (iii).

92. Important practical relations between longitudinal stress and strain.* — The important practical aspects of the relation between longitudinal stress and strain may best be brought out by considering the behavior of a steel rod (a test piece) which is subjected to a continually increasing longitudinal stress. Thus, the ordinates of the curves in Fig. 95 represent the values of an increasing longitudinal stress in pounds-weight per square inch, and the abscissas represent the corresponding elongations, in hundredths, of a rod of ordinary bridge steel.

Up to a point p, the position of which is not very sharply defined, the strain (elongation per unit initial length) is very exactly proportional to the stress (stretching force per unit of sectional area); that is the stress-strain curve is a straight line from O to p. Beyond the point p the stress-strain line is very slightly curved until the point q is reached where the steel begins to yield

* The student is referred to the splendid treatise on "The Materials of Construction" by J. B. Johnson, Wiley and Sons, 1898, for a full discussion of this subject.

very greatly. This yielding takes place rather irregularly until the whole test-piece has yielded, it alters the temper of the steel, and the steel then sustains an increased stress which reaches a maximum at the point *t*. The metal then begins to be weakened by the continued increase of length, and finally the rod breaks at the point *b*.

The point *p* marks the *true elastic limit;* the point *q*, which is sometimes called the *yield point*, marks what is for practical

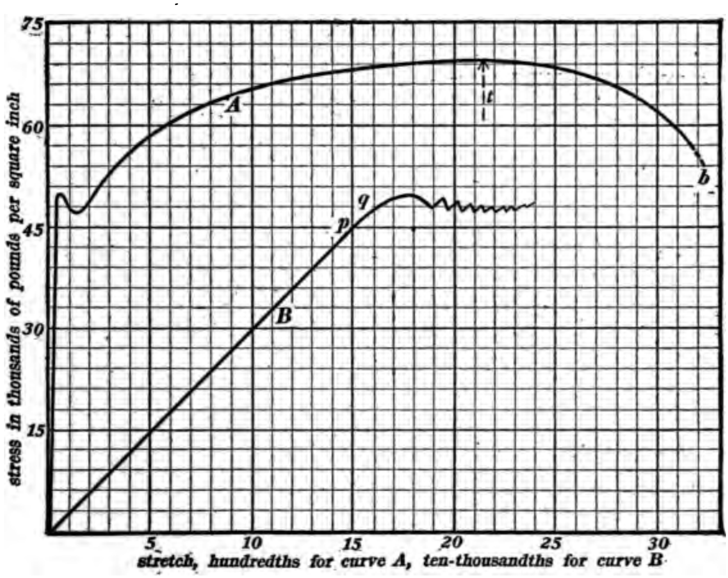


Fig. 95.

purposes* called the *elastic limit*, and the point *t* marks what is called the *tensile strength* of the steel. The most important of these points is the yield point or elastic limit, commercially so called, because the steel breaks after a very few applications of a stress that exceeds the elastic limit. Everyone is familiar with this fact in that a wire may be easily broken by bending it repeatedly beyond the elastic limit, and one can easily imagine how

* Because of the difficulty of locating the point *p* accurately.

short-lived a bridge or a steel rail would be if it were strained beyond the elastic limit by every passing train. The total elongation of a sample of steel at the breaking point *b*, Fig. 95, is an important indication of the toughness of the steel. The following table gives the important properties of several grades of steel.

TABLE.

Physical properties of steel.

Carbon content in per cent.	Elastic limit pounds per square inch.	Tensile strength pounds per square inch.	Elongation in 4 inches in per cent.	Stretch modulus in pounds per square inch.
0.17	51,000	68,000	33.5	29,800,000
0.55	57,000	106,100	16.2	"
0.82	63,000	142,250	8.5	"

93. Resilience. — The work done per unit volume in straining a substance to its elastic limit is called the *resilience* of the substance. This work per unit volume is equal to one half the product of the limiting stress and the limiting strain, according to equation (49), and it is represented by the area under the straight portion of the curve in Fig. 95. For example, the resilience of 0.82-per-cent.-carbon steel is

$$\tfrac{1}{2}\times 63{,}000\,\frac{\text{pounds}}{\text{inch}^2}\times 0.0021 = 66.1\,\frac{\text{inch-pounds}}{\text{inch}^3} = 5.5\,\frac{\text{foot-pounds}}{\text{inch}^3},$$

that is, 5.5 foot-pounds per cubic inch. Thus it would take 100 cubic inches of this steel (about 25 pounds) made into a spring to store sufficient energy to supply one horse-power for one second, provided the spring could be so designed as to be strained in every part to its elastic limit when wound up. The resilience of spring steel may be as high as 10 or 12 foot-pounds per cubic inch, the resilience of good cast iron is about 0.5 foot-pound per cubic inch.

The resilience of a substance is a measure of its strength to withstand a sudden shock, inasmuch as a blow of a hammer, for example, bends a bar until the kinetic energy of the hammer is all used in bending the bar. A structure subject to shocks should be made of highly resilient material.

94. Elastic hysteresis. — In nearly all substances there is more or less of a tendency for strain to persist after the stress has ceased. This is of course very markedly the case where a substance is strained beyond the elastic limit, but in many substances the elastic limit is by no means sharply defined, and very slight strains do not entirely disappear when the stress ceases. When a substance is subjected to a stress which increases and decreases periodically between two limiting values S_1 and S_2, the relation between stress and strain is somewhat as indicated in Fig. 96, where ordinates represent stress and abscissas represent strain. The branch *a* of the curve represents the relation between stress and strain while stress is increasing, and the branch *b* represents the relation between stress and strain while stress is decreasing. This divergence of the curve of stress and strain for increasing and decreasing stress is called *elastic hysteresis*. The increasing and decreasing stress is here supposed to increase and decrease very slowly. If the increase and decrease is rapid the divergence of the two curves *a* and *b*, Fig. 96, is due to hysteresis and also to elastic lag.

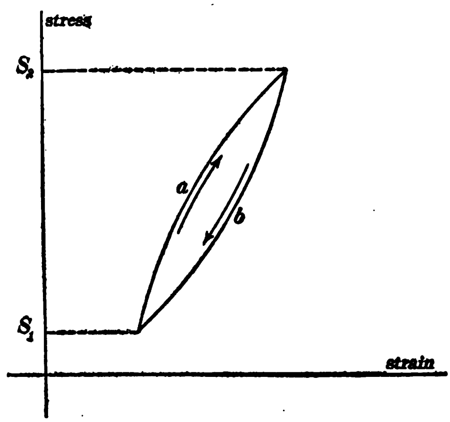


Fig. 96.

95. Elastic lag; viscosity. — Many substances, glass for example, when subjected to stress, take on a certain amount of strain quickly, after which the strain increases slowly for a time; and when the stress is relieved, a remnant of the strain persists for a time. This phenomenon is called *elastic lag*.

The strain of some substances, such as pitch, continues to increase indefinitely, although it may be very slowly, when they

are under stress. Such substances are said to be *viscous*. Nearly all metals are viscous when subjected to great stress.

Elastic hysteresis, elastic lag, and viscosity cause energy to be dissipated in a substance when it is strained. Thus the vibrations even of a steel spring die away rapidly in a vacuum, on account of the conversion of energy into heat as the spring is repeatedly distorted.

96. Elastic fatigue. — The repeated application of a stress weakens a metal so that it will break under less than its normal breaking stress, or less even than the stress corresponding to its

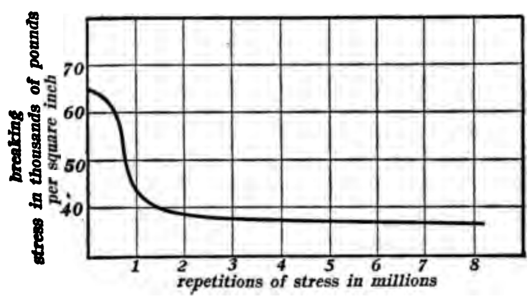


Fig. 97.

elastic limit. Fig. 97 shows the decease in tensile strength of a sample of mild steel with repetitions of the stress.

Continued repetition of stress causes an increase in the amount of energy dissipated by elastic lag and viscosity. Thus the vibrations of a torsion pendulum die away faster after it has been kept vibrating for several days, than at first.

HYDROSTATIC PRESSURE AND ISOTROPIC STRAIN.

97. Hydrostatic pressure.* — A fluid at rest not only pushes normally against a surface which is exposed to its action, but two contiguous portions of a fluid at rest always push on each other at right angles to a small plane q which may be imagined to separate them as indicated in Fig. 98. Whatever the *direction* of the small plane q may be, the *force action per unit area* across

* Hydrostatic pressure is discussed also in the chapter on hydrostatics.

it is the same. This fact was first pointed out by Pascal (1623–1662) and it is sometimes called Pascal's principle.* The force action per unit area at a point in a fluid is generally represented by the letter p and it is called the *hydrostatic pressure* at the point.

98. Isotropic strain. — When a substance like glass or cast metal is subjected to an increase of hydrostatic pressure the substance is reduced in size without being changed in shape; such a strain is called an *isotropic strain.* Let V be the original volume of the substance, and let v be the diminution of volume due to the increase of pressure. It is convenient to express v as a fraction of V, and this fraction v/V is used as a measure of the isotropic strain.

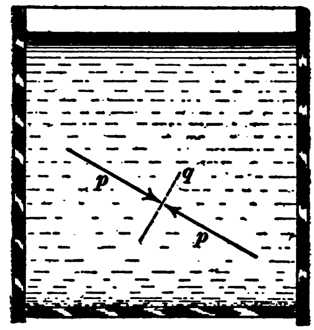


Fig. 98.

99. Bulk modulus of a substance. — The diminution of volume of a substance per unit of original volume (v/V) is proportional to the increase of hydrostatic pressure, except, of course, when the increase of pressure is very great and the decrease of volume considerable. Therefore, for small changes of volume, the ratio of increase of hydrostatic pressure to decrease of volume per unit volume is a constant for a given substance (at a given temperature). That is

$$B = \frac{p}{\frac{v}{V}} = \frac{pV}{v} \tag{51a}$$

in which V is the volume of a substance at a given pressure, v is the decrease of volume due to an increase of pressure p, and B is a constant for the given substance. This quantity B is called the *bulk modulus* of the substance. The reciprocal of B is called

* The student should attempt to establish this principle by considering the equilibrium of a prism shaped portion of a fluid at rest.

the *coefficient of compressibility* of the substance. Therefore, writing C for $1/B$ equation (51)a becomes

$$C = \frac{v}{pV} \tag{51b}$$

or

$$v = pVC \tag{51c}$$

The coefficient of compressibility of a substance is the change of volume per unit original volume per unit increase of pressure, and, according to equation (51)c, the decrease of volume of a substance due to a given increase of pressure is equal to the product of the increase of pressure, the original volume, and the coefficient of compressibility.

TABLE.*

Coefficients of compressibility at 20° C. for moderate increase of pressure.

(Decrease of volume per unit volume per atmosphere increase of pressure.)

SUBSTANCE.	$C \times 10^6$.
Ether	170.0
Alcohol	101.0
Water	46.0
Glass	2.2
Steel	0.68

100. Potential energy of isotropic strain.—The potential energy per unit volume of an elastic substance under increased pressure is equal to one half the product of the increase of hydrostatic pressure p and the strain v/V, or it is equal to one half the product of the bulk modulus B and the square of the strain (v^2/V^2). This relation may be derived in a manner very similar to the proof of equations (49) and (50).

101. Compressibility of gases. Boyle's law.—Solids and liquids generally decrease but slightly in volume when subjected to increase of pressure. Thus the volume of water decreases about one ten-thousandth part when subjected to an increase of

*See *Physikalish-Chemische Tabellen* by Landolt and Börnstein, Berlin, 1895, for a very complete collection of data of all kinds.

pressure oi 30 pounds per square inch, and the volume of steel decreases about one ten-thousandth part when subjected to an increase of pressure of 2,000 pounds per square inch.

Gases, on the other hand, decrease greatly in volume when subjected to increase of pressure. The remarkable contrast between water and air in regard to compressibility may be shown by filling a bicycle pump with air and then with water, and striking the piston rod in each case with a hammer. The air will be found to act as a cushion and the water will appear to be as solid as if the whole pump barrel and piston were one piece of steel. When a steam engine is started, the water which usually collects in the steam pipes may enter the cylinder in sufficient quantity to cause the moving piston to burst the cylinder head.

When the temperature of a gas is kept at a constant value, the volume of the gas is inversely proportional to the pressure to which the gas is subjected. That is

$$v = \frac{k}{p}$$

or

$$pv = k \tag{52}$$

in which v is the volume of a given amount of gas, p is the pressure of the gas, and k is a constant. This relation, which is known as Boyle's law, was discovered by Robert Boyle's * in 1660, and more completely established by Mariotte, who discovered it independently in 1676. It is very accurately true of such gases as hydrogen, nitrogen and oxygen at ordinary temperatures and pressures, but all gases deviate from it appreciably, especially at low temperatures and under great pressures. See the discussion of the properties of gases in the chapters on heat.

SHEARING STRESS AND SHEARING STRAIN.

102. Shearing stress. — The type of stress and strain in a twisted rod is called shearing stress and shearing strain, and the discussion of this type of stress and strain is somewhat obscured by the fact that there is no familiar example in which homogeneous shearing stress and shearing strain occur; the stress and strain in a twisted

* *New Experiments touching the Spring of Air*, Oxford, 1660.

rod are non-homogeneous. Any intelligent discussion of shearing stress and shearing strain must, however, be based on a case in which the stress and strain are homogeneous. Consider, therefore, a cubical portion of a substance $ABCD$, Fig. 99, and suppose that outward forces (S units of force per unit of area) act upon the faces AB and CD, that inward forces (S units of force per unit of area) act upon the faces AC and BD, and that no force at all acts on the two faces of the cube which are parallel to the plane of the paper. Then the material of the cube will be subject to what is called a *shearing stress*, and the stress will be homogeneous. The character

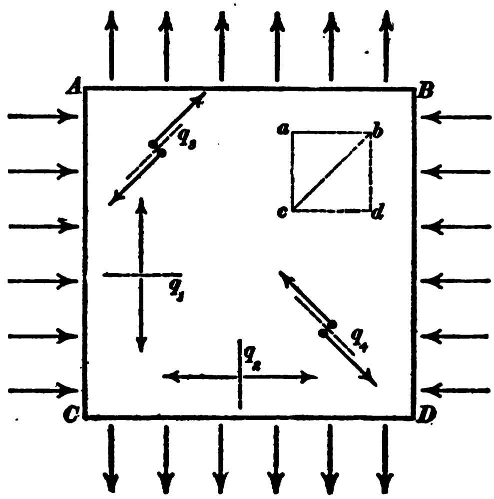


Fig. 99.

of the force action between contiguous parts of the material of the cube is as follows: A *pull* of S units of force per unit area acts across any plane q_1 which is parallel to the faces AB and CD of the cube; a *push* of S units of force per unit area acts across any plane q_2 which is parallel to the faces AC and BD of the cube; a sliding force, or tangential force, as it is called, of S units of force per unit of area acts across any plane q_3 or q_4 which is parallel to the diagonal plane AD or BC; and no force at all acts across a plane which is parallel to the plane of the paper.

To show that the force action across the diagonal planes q_3 and q_4 is a tangential force action and that the tangential or sliding force is S units of force per unit area, consider any unit cube $abcd$ of the material. The area of each face of this cube is unity, and the area of the diagonal plane bc is $\sqrt{2}$ units. The total force acting on the face bd is a push of S units, the total force acting on the face cd is a pull of S units, and the resultant of these two forces is a force parallel to bc and equal to $\sqrt{2}S$ as shown in Fig. 100. Similarly, the resultant of the forces acting on the faces ab and ac is a force parallel to cb and equal to $\sqrt{2}S$. Therefore, the force action across the diagonal plane bc is a tangential force action equal to $\sqrt{2}S$, which, divided by the area of the diagonal plane bc, gives a tangential force action of S units of force per unit of area.

It can be shown in the same way that the force action across q_4, Fig. 99, is a tangential force of S units per unit of area. It is on account of the purely tangential forces across q_3 and q_4 that this type of stress is called a shearing stress.

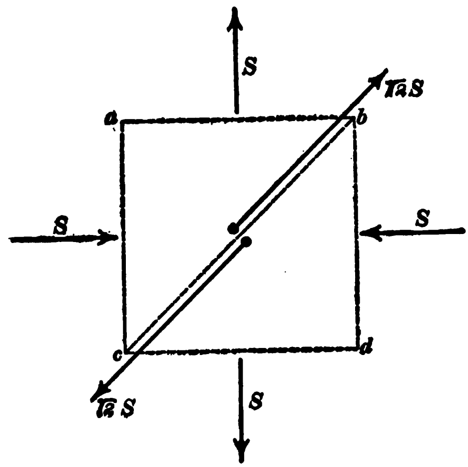


Fig. 100.

108. Shearing strain.—The effect of the prescribed stress in Fig. 99 is to shorten the cube in the direction of the push and to lengthen the cube by an equal

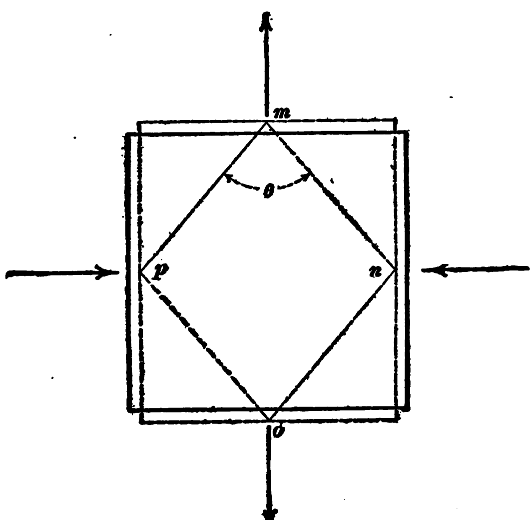


Fig. 101.

amount in the direction of the pull, without changing the dimensions of the cube in a direction at right angles to the plane of the paper in Fig. 99. This distortion is represented by the dotted rectangle in Fig. 101, and the dotted rhombus *mnop* in Fig. 101 is a figure which was square in the unstrained material. The angle θ is less than $90°$ and the value of the angle $90° - \theta$ (expressed as a fraction of a radian) is called the *angle of the shearing strain*, or simply the *angle of shear*. It is usually represented by the letter ϕ, and it is used as a measure of the shearing strain.

Let L be the original length of each edge of the cube in Fig. 99, and let l be the increase of length in the direction of the pull and the decrease of length in the direction of the push. It is convenient to express l as a fraction of L and we will represent this fraction by the letter α $(= l/L)$. It is important to know that the angle of shear ϕ as above defined is equal to 2α. That is

$$\phi = 2\alpha \tag{53}$$

The full-line square in Fig. 102 is the figure which when distorted becomes the rhombus in Fig. 101, and the small triangle in Fig. 102 is enlarged in Fig. 103. The

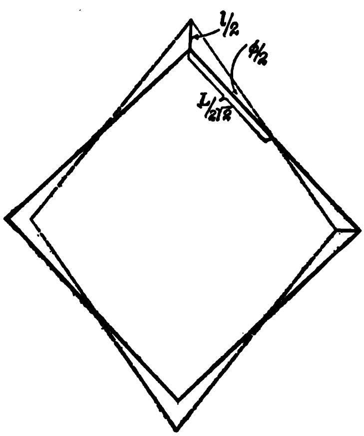


Fig. 102.

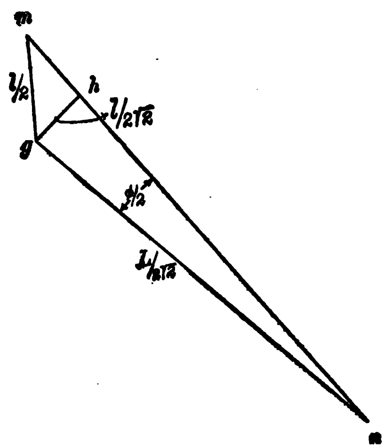


Fig. 103.

angle $\phi/2$ is very small and it is therefore sensibly equal to gh divided by gn. That is, $\phi/2$ is equal to l/L $(= \alpha)$ so that $\phi = 2\alpha$.

104. Slide modulus of a substance. — The angle of shear ϕ produced in Fig. 99 by the shearing stress S is proportional to S, according to Hooke's law, so that the ratio S/ϕ is a constant for a given substance, within the limits of elasticity. This constant is called the *slide modulus* of the substance and it is represented by the letter n. Therefore we have

$$n = \frac{S}{\phi} \tag{54}$$

The slide modulus of a substance is sometimes called the *shearing modulus* of the

substance. It is approximately equal to $\frac{2}{5}$ of the stretch modulus (Young's modulus) for metals.

105. Discussion of a twisted rod. — Consider a cylindrical rod of radius R and length L, and suppose that one end of the rod is fixed while the other is turned through the angle θ so as to twist the rod. Consider a cylindrical shell of the material of the rod of which the radius is r, and imagine this cylindrical shell to be cut along one side and laid out flat so that it may be pictured on a flat surface.* Figures 104

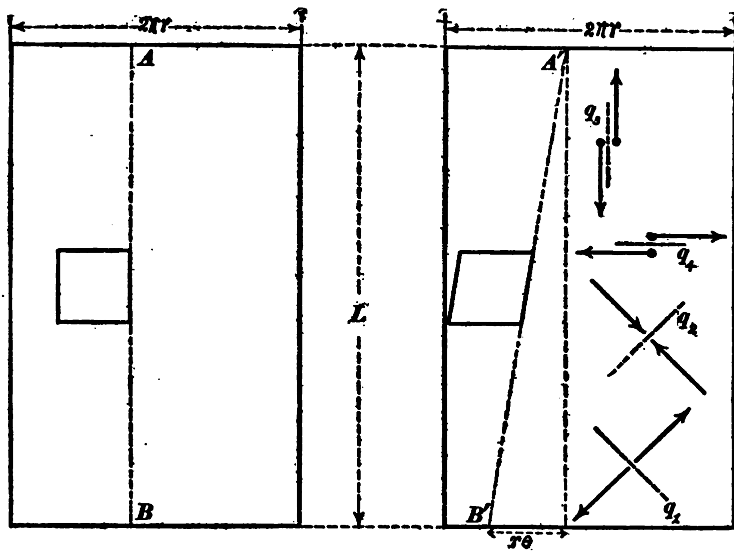


Fig. 104. Fig. 105.

and 105 represent the cylindrical shell laid out flat, or developed, in this way; Fig. 104 before twisting, and Fig. 105 after twisting. The line AB which is parallel to the axis of the rod in the untwisted rod takes the position $A'B'$ after the rod is twisted, and the small square becomes a rhombus. The portions of the rod in the cylindrical shell under consideration are subjected to a shearing strain of which the angle of shear is

$$\phi = \frac{r\theta}{L} \tag{55}$$

* This discussion of the strain in a twisted rod may be made more easily intelligible by means of a model as follows: A tin cylinder about 20 cm. in diameter and 35 cm. long, has wooden disks fixed in each end. On one end is an additional wooden disk which turns on a nail which is in the axis of the cylinder. Loosely woven muslin is tacked at one end to the movable disk and at the other end to the fixed disk. This muslin fits the tin cylinder closely, and the seam at one side is sewed. On this muslin a small square may be drawn like Fig. 104, and also small circles, and when the movable disk is turned through a considerable angle, the distortion of the square and of the circles will give a clear idea of the character of the strain in a cylindrical shell of a twisted rod.

as is evident from Fig. 105. The character of the force action between contiguous portions of the material of the rod may be understood by comparing Fig. 105 with Fig. 99. There is a tangential force action across every vertical plane q_3 and across every horizontal plane q_4,* there is a normal pull across every plane like q_1, and a normal push across every plane like q_2; and the force per unit area, S, in each case is equal to $n\phi$, according to equation (54). Therefore, substituting for ϕ its value from equation (55), we have

$$S=\frac{nr\theta}{L}. \tag{56}$$

The tangential stress across vertical planes like q_3, Fig. 105, is concentrated at the bottom of a sharp groove cut in the rod parallel to its axis, like a key seat in a shaft, and such a groove therefore weakens the rod very much indeed.

Constant of torsion of a rod or wire. — The total force action across a complete section of a twisted rod is a torque T about the axis of the rod and the value of the torque is

$$T=-\frac{\pi n R^4\theta}{2L} \tag{57}$$

in which n is the slide modulus of the material of the rod or wire, R is the radius of the rod or wire, L is the length of the rod or wire, and θ is the angle through which one end of the rod or wire is twisted. The negative sign is written for the reason that the torque tends to reduce θ,† that is, T and θ are opposite in sign. This equation (57) shows that T is proportional to θ, and the proportionality factor $\pi nR^4/2L$ is called the *constant of torsion* of the wire or rod. If the constant of torsion of a rod or wire is determined by observing the angle of twist θ produced by a known torque, the slide modulus of the material may be calculated from equation (57).

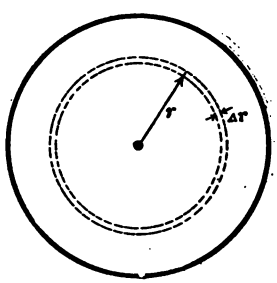


Fig. 106.

Proof of equation (57). — Let Fig. 106 represent a sectional view of the rod. Consider the narrow annulus of width Δr and radius r, as shown by the dotted lines. The force action per unit area across this annulus is $rn\theta/L$ according to equation (56), and this force action is at right angles to r at each point of the annulus. Therefore the torque action, about the

* The force actions between contiguous portions of a twisted rod are inferred from the character of the distortion at each point as represented in Fig. 105. These force actions may be made clearly evident by the use of two models, as follows: (a) A bundle of smooth square pine sticks bound together and, if desired, turned to a cylindrical shape, shows sliding along vertical planes like q_3, Fig. 105, when the bundle of sticks is twisted. (b) A brass tube with a slit along a helical line which is at each point inclined at an angle of 45° to the axis of the tube, is as strong as an uncut tube to withstand a twist in one direction (the faces of the slit push against each other normally), but the slit opens when the tube is twisted in the other direction.

† The reacting torque of the twisted rod is here referred to.

axis of the rod, of the force which acts across unit area of the annulus is $r \times rn\theta / L$, which, multiplied by the area $2\pi r \cdot \Delta r$ of the annulus, gives the total torque action ΔT across the annulus. That is

$$\Delta T = \frac{2\pi r^3 n\theta}{L} \cdot \Delta r$$

or

$$T = \frac{2\pi n\theta}{L} \int_0^R r^3 dr = \frac{\pi n R^4 \theta}{2L}$$

GENERAL EQUATIONS OF STRESS AND STRAIN.

106. Principle stretches of a strain. — A small spherical portion of a body always becomes an ellipsoid when the body is distorted. A distortion which changes a sphere into an ellipsoid consists always of simple increase or decrease of linear dimensions in three mutually perpendicular directions. These mutually perpendicular directions are called the *axes of the strain*, and the increase of length per unit length (l/L) in the directions of the respective axes are called the *principal stretches* of the strain. The principal stretches of a strain are represented by the letters f, g, and h.

107. Principal pulls of a stress. — Imagine a small plane area q, called a *section*, in the interior of a body under stress. The portions of the body on the two sides of this section exert on each other a definite force in a definite direction, and the force per unit area of the section is called the *stress on the section*. When the force is normal to the section, the stress on the section is called a *pull*, positive or negative as the case may be. When the force is parallel to the section, the stress on the section is called a *tangential stress*.

The conditions of equilibrium of a small portion of a body under stress require* that at a point in the body there be three mutually perpendicular sections across which the force action is normal, or on which the stress is a pull. These three sections are called the *principal sections of the stress*, the three lines perpendicular to them are called the *axes of the stress*, and the pulls on the three sections are called the *principal pulls of the stress*. These three pulls constitute the stress at the point and if these pulls are specified the stress is completely determined.

108. General equations of stress and strain. — Let F be a longitudinal stress, that is, a simple pull, in the direction of the x-axis of reference. Such a stress causes a stretch aF in the direction of the x-axis, and a negative stretch $-bF$ in the directions of the y and z-axes. Therefore, writing f', g' and h' for the three stretches due to the simple pull F, we have

$$\begin{aligned} f' &= aF \\ g' &= -bF \\ h' &= -bF \end{aligned} \tag{i}$$

Similarly, let G be a longitudinal stress in the direction of the y-axis of reference, and let f'', g'', and h'' be the three stretches, parallel to x, y, and z axes respectively, produced by G. Then we have:

$$\begin{aligned} f'' &= -bG \\ g'' &= aG \\ h'' &= -bG \end{aligned} \tag{ii}$$

* See *Elasticity, theory of*, in Encyclopedia Britannica, 9th Ed.

Similarly, let H be a longitudinal stress parallel to the z-axis of reference, and let f''', g''', and h''' be the three stretches, parallel to the x, y, and z axes, respectively, produced by H. Then we have:

$$\begin{aligned} f''' &= -bH \\ g''' &= -bH \\ h''' &= aH \end{aligned} \qquad \text{(iii)}$$

Experiment shows* that the stretch produced in any direction by a number of pulls acting together is equal to the sum of the stretches in that direction produced by the respective pulls acting separately. Therefore:

$$\begin{aligned} f &= f' + f'' + f''' \\ g &= g' + g'' + g''' \\ h &= h' + h'' + h''' \end{aligned} \qquad \text{(iv)}$$

in which f, g, and h are the stretches, parallel to the x, y, and z axes, respectively, produced by the three pulls F, G, and H acting together. Therefore substituting the values of f', f'', f''', g', g'', g''', h', h'', and h''' from equations (i), (ii), and (iii) in equation (iv), and we have:

$$\begin{aligned} f &= aF - bG - bH \\ g &= -bF + aG - bH \\ h &= -bF - bG + aH \end{aligned} \qquad (58)$$

These equations give the strain (f, g, and h) which is produced in an isotropic elastic solid by any stress (F, G, and H), and it shows that an isotropic elastic solid has but two constants of elasticity a and b. In fact the quantity a is equal to $1/E$ and the quantity b is equal to σa, where σ is the value, β'/β, of Poisson's ratio. See Art. 87. Starting with these relations, it is very easy to derive expressions for the bulk modulus B and for the slide modulus n of a substance in terms of E and σ by using equations (58); considering that $F = G = H = p$ in the case of hydrostatic pressure, and that $F = +S$, $G = -S$, and $H = 0$ in the case of a shearing stress.

PROBLEMS.

109. A helical spring is elongated by an amount of 1.2 inches when a 4-pound weight is hung upon it. How much additional elongation is produced by 1 pound additional weight? By two

* In general, any effect which is *proportional* to a cause, may be resolved into parts which correspond to the parts of the cause. Thus a spring stretches in proportion to the stretching force. One kilogram produces, say, one centimeter elongation; two kilograms produce two centimeters elongation, which is one centimeter for each kilogram. See footnote to Art. 32.

pounds additional weight? By three pounds additional weight? By four pounds additional weight?

Note.—Assume in this problem, and in those that follow, that the elastic limit is not exceeded.

110. The middle of a long beam is depressed 2 inches by a load of 5,000 pounds. How much will it be depressed by a load of 15,000 pounds?

111. A long rod is fixed at one end, and a twisting force, or torque, of 100 pound-inches applied at the free end causes the free end to turn through an angle of 10°. What torque would be required to turn the free end of the rod through 26°?

112. A rod 2 inches in diameter and 20 feet long is stretched to a length of 20 feet and ¼ inch by a force of 10,000 pounds-weight. What is the value of the longitudinal stress, and what is the value of the longitudinal strain?

113. A rod 20 feet long and 1 inch in diameter is subjected to a pull of 20,000 pounds per unit of sectional area causing it to be lengthened to 20.02 feet, that is one part in a thousand, and causing it to contract to a diameter of 0.997 inch, that is, three parts in ten thousand. What is the length and what is the diameter of the rod when it is subjected to a pull of 40,000 pounds per unit sectional area?

114. A wire 200 inches long and 0.1 inch in diameter is pulled with a force of 150 pounds. The elongation produced is ½ inch. What is the value of the stretch modulus of the material?

115. A wire five feet long and 0.06 square inch sectional area is subjected to a stretching force of 300 pounds. The stretch modulus of the material is 28,000,000 pounds per square inch. What elongation is produced?

116. A steel beam is bent so that its middle line forms the arc of a circle 600 inches in radius. What is the elongation per unit length of a filament 2 inches from the middle line?

117. The stretch modulus of the steel of which the beam of the previous problem is made is 30,000,000 pounds per square inch.

What is the pull (force per unit area of course) of a filament of the beam 2 inches from the middle line of the beam?

118. What is the resilience of spring steel of which the elastic limit is 70,000 pounds per square inch and of which the stretch modulus is 30,000,000 pounds per square inch?

119. A cork $\frac{1}{2}$ inch in diameter is pushed with a force of 20 pounds-weight into a bottle which is completely filled with water. What hydrostatic pressure is produced in the bottle? Neglect the friction of the cork against the glass neck of the bottle.

120. A body subjected to hydrostatic pressure is decreased in length, in breadth, and in thickness by 5 parts in a thousand (initial). By how many parts per thousand (initial) is the volume reduced?

121. 2,000 cubic inches of water are reduced to 1,880 cubic inches by an hydrostatic pressure of 3,000 lbs. per square inch. What is the value of the bulk modulus of water?

122. Calculate the compressibility of water from the answer to the previous problem and explain its meaning.

123. A bicycle pump is full of air at 15 pounds per square inch, length of stroke is 12 inches; at what part of the stroke does air begin to enter the tire at 40 pounds per square inch above atmospheric pressure? Assume the compression to take place without rise of temperature.

124. The clearance space behind the piston of an air compressor when the piston is at the end of its stroke is $\frac{1}{50}$ of the *volume swept by piston* during the stroke. What is the greatest pressure that can be produced in a compressed air reservoir by this compressor, the compression of the air in the cylinder being assumed to be without change of temperature?

Note.—As a matter of fact the air in an air compressor is heated very considerably by the compression.

125. The piston of an air pump is 0.01 inch from the bottom of the cylinder when it is at the end of its stroke, and the pressure of the air in the clearance space is then at atmospheric pressure.

The length of stroke is 6 inches. What is the lowest vacuum which can be produced by the pump?

126. A cubical piece of steel is shortened two parts in a thousand in one direction, lengthened two parts in a thousand in a direction at right angles to the first, and unchanged in dimension in the third direction, as represented in Fig. 101. What is the value of the angle of shear in degrees?

127. A steel rod 120 inches long is fixed at one end and the other end is turned through 5 degrees of angle. Consider a small portion p of the rod at a distance of 1 inch from the axis of the rod. Find the angle of shear ϕ of this small portion p of the metal.

128. The slide modulus of the steel used in the rod of problem 127 is 12 million pounds per square inch. Find the shearing stress in the small portion p of the rod.

129. A steel shaft 500 inches long and 3 inches in diameter transmitting 100 horse-power is subject to a torque of 23,100 pound-inches of torque. The slide modulus of the material of the shaft is 12,000,000 pounds per square inch. Calculate the angle through which one end of the shaft is twisted relative to the other end.

130. The three stretches of a strain are $+0.015$, $+0.025$ and -0.025. What are the semi-axes of the ellipsoid into which a sphere 10 inches in radius is distorted by this strain? Strain supposed to be homogeneous.

131. A force of 250 pounds acts across a section of which the area is $\frac{1}{4}$ square inch. What is the value of the stress on the section?

132. A square rod $2 \times 1\frac{1}{2}$ inches is subjected to a tension of 75,000 pounds. What kind of stress acts across a section of the rod and what is its value?

133. Two long strips of metal are lapped and fastened by a single rivet of which the sectional area is two square inches. The two strips are subjected to a tension of 10,000 pounds. What kind of stress acts across the middle section of the rivet and what is the value of the stress?

134. Derive the equation expressing the bulk modulus of a substance in terms of its stretch modulus and Poisson's ratio. The stretch modulus of steel is 30 million pounds per square inch, and Poisson's ratio for steel is 0.28. Find the value of the bulk modulus, and find the value of the coefficient of compressibility and compare it with the value given in the table in Art. 99. One atmosphere is equal to 14.7 pounds per square inch.

135. Derive the equation expressing the slide modulus of a substance in terms of its stretch modulus and Poisson's ratio, and calculate the slide modulus of steel using the data given in problem 134.

CHAPTER IX.

HYDROSTATICS.

109. Pressure at a point in a fluid.* — The force with which a fluid at rest pushes against an element of an exposed surface is at right angles to the element and proportional to the area of the element. The *force per unit area* is called the *hydrostatic pressure* or simply the *pressure* of the fluid at the place where the element of area is located and it is usually represented by the letter p. When the pressure has the same value throughout a fluid the pressure is said to be uniform, when the pressure varies from point to point in a fluid the pressure is said to be non-uniform. When the pressure in a fluid is uniform the total force F acting on an exposed plane surface is

$$F = pa \tag{59}$$

where a is the area of the surface.

Examples. (a) *Steam pressure.* — The piston of a steam engine is pushed by a force equal to pa, where a is the area of the piston, and p is the pressure of the steam in the cylinder. Every part of the inside surface of a steam boiler is pushed outwards by the steam.

(b) *Atmospheric pressure.* — The force with which the air pushes on the surfaces of bodies does not ordinarily appeal to our senses. It is shown however by the collapse of a thin-walled vessel when the inside pressure is reduced by pumping out the air. Atmospheric pressure is also strikingly shown by means of the apparatus known as the *Magdeburg Hemispheres.* This consists of two metal cups which fit together air tight and form a hollow vessel from which the air may be removed by pumping.

* See Art. 97.

The pressure of the outside air then holds the cups together and a considerable effort is required to separate them. This celebrated experiment was devised by Otto von Guerike, the inventor of the air pump, and it was performed publicly in Magdeburg in 1654.

(c) *The hydrostatic press* consists essentially of a strong cylinder with a large plunger or piston, and a pump with a small piston or plunger for forcing water into the large cylinder under high pressure. The great forging press at the Bethlehem Steel Works has two plungers each fifty inches in diameter, thus exposing a total of about 3,600 square inches of piston area to the water, which is forced into the cylinders of the press at a pressure of 8,000 pounds per square inch. This gives a total force of about 14,000 tons upon the two plungers.

110. The circumferential tension in the walls of a cylindrical pipe. — The pressure of a fluid in a cylindrical pipe produces a

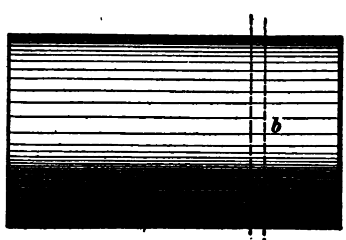


Fig. 107.

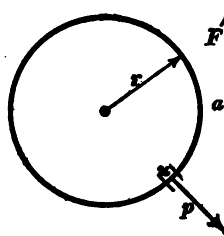


Fig. 108.

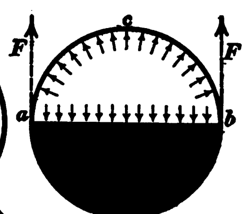


Fig. 109.

tension in the material of the pipe. Consider a narrow band b, Fig. 107, of the material of a pipe, the width of the band being one inch. An end view of this band is shown in Fig. 108, and, since the band is one inch wide, each inch of its circumference is pushed outwards by a force equal to p pounds, where p is the steam or water pressure in pounds per square inch. Therefore, according to Art. 60, the circumferential tension in the band b is equal to rp pounds per inch of width, where r is the radius of the pipe in inches.

It is instructive to establish this result from another point of view as follows: Imagine the cylindrical pipe to be half solid, as shown by the shaded area in Fig. 109, then, considering one inch of length of the pipe as before, the area of the flat surface ab is $2r$, the force acting on this flat face is $2rp$, and this force is balanced by the two forces FF, so that the value of each force F is equal to rp.

Since the circumferential tension in a cylindrical pipe is equal to rp, it is evident that a small pipe can withstand a much greater pressure than a large pipe, the thickness of the walls of the pipe being the same.

111. Pressure in a liquid due to gravity. — The pressure in a fluid under the action of gravity increases with the depth. If the density of the fluid is the same throughout, and this is approximately the case in any liquid, then the pressure at a point distant x centimeters beneath the surface of the liquid *exceeds the pressure at the surface* by the amount

$$p = xdg \tag{60}$$

in which p is expressed in dynes per square centimeter, d is the density of the liquid in grams per cubic centimeter, and g is the acceleration of gravity in centimeters per second per second. If the factor g is omitted, this equation gives the value of p in grams-weight per square centimeter.

The pressure at a point x feet beneath the surface of water exceeds the pressure at the surface by the amount $p = 0.434x$, where p is expressed in pounds-weight per square inch. The factor 0.434 is the weight in pounds of a prism of water one foot long and one square inch base, that is, this factor is the density of water in pounds per inch^2-foot, and when multiplied by feet it gives pounds per square inch.

Discussion of equation (60). — The force with which a liquid acts on an element of an exposed surface is independent of the direction of the element.* Therefore we may derive equation

* See the discussion of Pascal's principle in Art. 97.

(60) by considering a horizontal surface of area a exposed to the action of the liquid as shown in Fig. 110. The volume of liquid directly above a is ax, the mass of this portion of liquid is axd, the force in dynes with which gravity pulls on this portion of liquid is $axdg$, and therefore the total force with which this portion of liquid pushes down on the element a is equal to $axdg$ dynes, so that the force per unit area is $axdg$ divided by a.

Equation (60) involves no consideration of the shape of the vessel which contains the liquid. As a matter of fact the pressure at a point in a liquid exceeds the pressure at the surface of the liquid by the amount xdg whatever the shape and size of

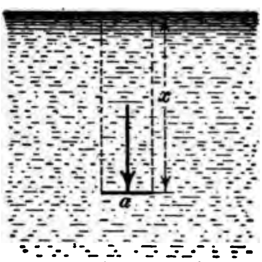


Fig. 110.

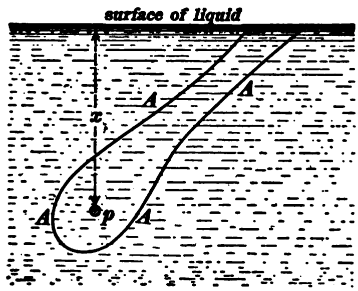


Fig. 111.

the containing vessel may be. This may be made almost self-evident as follows: Given a point p, Fig. 111, at a distance x beneath the surface of a large body of liquid. *Imagine a portion of the liquid* $AAAA$, *of any shape whatever, extending from* p *to the surface.* The liquid surrounding the portion $AAAA$ acts on $AAAA$ exactly as a containing vessel of the same shape would act, and therefore the pressure of p is exactly what it would be if the portion $AAAA$ were contained in such a vessel.

112. The total force acting on a water gate and its point of application. — When a plane surface of area a is exposed to the action of a fluid under uniform pressure, the total force acting on the surface is pa and the point of application of this force is the center of figure of the exposed plane surface. When, however,

a plane surface is exposed to the action of a fluid in which the pressure is not uniform, the total force is, of course, *not* equal to pa, for p has different values at different parts of the surface,

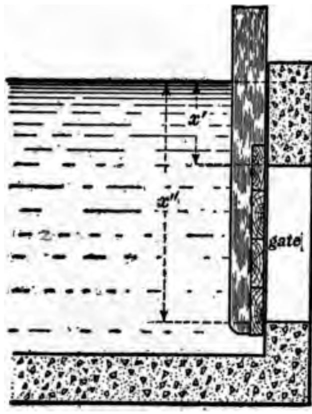


Fig. 112*a*.

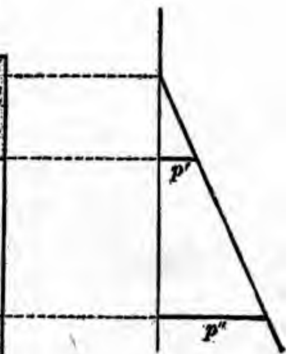


Fig. 112*b*.

and the point of application of the total force is *not* at the center of figure of the exposed surface. The simplest case is that in which the water in a tank pushes against the rectangular side of the tank, or the case in which water pushes against a rectangular gate as shown in, Fig. 112*a*. The pressure at the top of the gate is $p' = 0.434x'$ pounds per square inch, the pressure at the bottom of the gate is $p'' = 0.434x''$, the average pressure over the whole gate is $(p' + p'')/2$ or $0.434(x' + x'')/2$ pounds per square inch, and the total force F acting on the gate is equal to the product of this average pressure and the area of the gate in square inches.

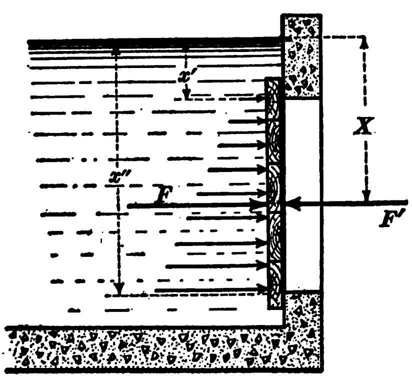


Fig. 113.

The point of application of the total force with which the water pushes on the gate is the point at which a single force F', Fig. 113, could be applied to balance the push of the water. This point is evidently below the center of the gate; in fact the distance X, Fig. 113, is

$$X = \frac{2}{3}\left(\frac{x''^3 - x'^3}{x''^2 - x'^2}\right) \tag{61}$$

In the case of the side of a rectangular tank, or in case of a dam (where x' equals zero) the distance from the surface of the water to the point of application of the total force which pushes on the side of the tank or against the dam is two thirds of the depth of the water.

Proof of equation (61).—The total force F, Fig. 113, is equal to the area of the gate $w(x'' - x')$ multiplied by the average pressure $0.434(x'' + x')/2$, w being the horizontal width of the gate; and the torque action of F about any conveniently chosen point is equal to the sum of the torque actions, about the same point, of the forces acting on the various elements of the surface of the gate. Consider a horizontal strip of the gate distant x beneath the surface of the water and of which the vertical breadth is dx. The force acting on this strip is $0.434x \times wdx$, and the torque action of this force about a point at the surface of the water (lever arm x) is $0.434x^2 \times wdx$. Therefore the total torque action, about the chosen point, of the forces acting on the gate is equal to

$$0.434w\int_{x'}^{x''} x^2dx = 0.434w \times \tfrac{1}{3}(x''^3 - x'^3)$$

whence, placing this equal to the torque action $XF[=X \times 0.434w(x''^2 - x'^2) \times \frac{1}{2}]$, we have equation (61).

113. Measurement of pressure. The barometer.—The barometer consists of a glass tube T, Fig. 114, filled with mercury and inverted in an open vessel of mercury CC, the tube being of such length that an empty space V is left in which the pressure is zero.* The pressure in the tube at the level of the mercury in the open vessel is equal to atmospheric pressure, and it exceeds the pressure in the region V by the amount xdg according to equation (60). Therefore, since the pressure in V is zero, the

* Even if the tube is filled with extreme care so as to exclude all of the air, mercury vapor will form in the region V and the pressure will not be exactly zero.

value of atmospheric pressure is equal to xdg. This expression gives the value of atmospheric pressure in dynes per square centimeter, x being in centimeters, d being the density of the mercury in grams per cubic centimeter, and g being the acceleration of gravity in centimeters per second per second.

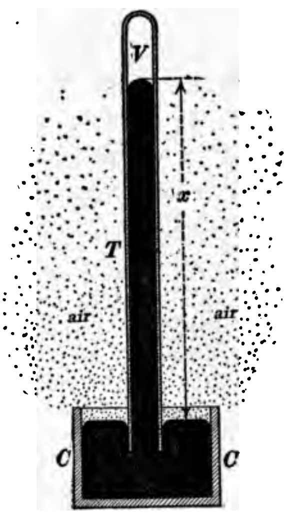


Fig. 114.

If the mercury is at some standard temperature, d is invariable; and if the barometer is used in a given locality, g is invariable; and *under these conditions the distance x may be used as a measure of the pressure.* In fact, atmospheric pressure is usually expressed in terms of the height the barometric column would have in millimeters or in inches if the mercury were at 0° C. and if the value of the acceleration of gravity were 981.61 cm./sec^2 (its value at 45° north latitude at sea level). To facilitate the accurate use of the barometer in different localities and at different temperatures, tables * have been published, with the help of which the height of barometric column under standard conditions as to temperature and gravity may be easily found from its observed height under known conditions.

114. Measurement of pressure. Manometers or pressure gauges. — The barometer is used for the measurement of the pressure of the atmosphere. An instrument for measuring the difference between the pressure in a closed vessel and atmospheric pressure is called a *manometer* or a *pressure gauge.*

The open tube manometer. — When the pressure to be measured is small, for example, when it is desired to measure the

* To be found in many laboratory reference books. For example, in Kohlrausch's *Physical Measurements*, and in Landolt and Börnstein's *Physikalisch-Chemische Tabellen.*

pressure of the gases at the base of a smoke-stack, or the pressure developed by a fan blower, the pressure is determined by measuring the height of water or mercury column which it will support. Thus Fig. 115 shows an open tube manometer arranged for measuring the pressure of the gas in city mains.

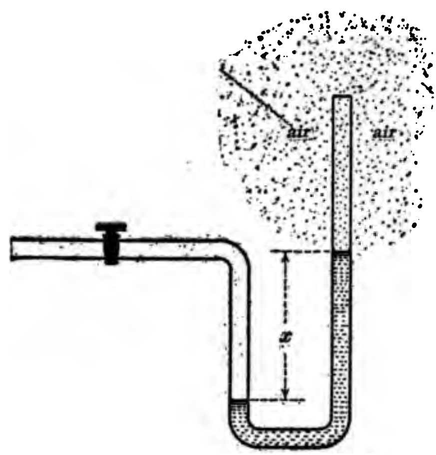


Fig. 115.

The Bourdon gauge. — The pressure gauge commonly used on steam boilers is usually of the type known as the Bourdon gauge, of which the essential features are shown in Fig. 116. A very thin walled metal tube *abc* of flat elliptical section is closed at the end *c*, and the end *a* communicates through the tube *tt* with the steam boiler. The pressure inside

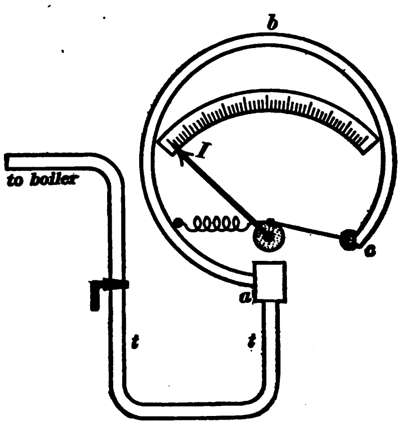


Fig. 116.

Fig. 117.

of the tube *abc* tends to straighten it, and the movement of the end *c* actuates a pointer which plays over a scale the divisions

of which are determined by calibration, that is, by noting the position of the pointer for various known pressures.

The gauge tester is a device for generating accurately known pressures which are communicated to a pressure gauge which is to be calibrated. It consists of a small metal chamber filled with oil. A plunger of known area a is forced into this chamber by a known weight, and the known pressure thus developed is communicated to the gauge.

115. Buoyant force of fluids. — A body which is submerged in a fluid is pushed upwards by a force which is equal to the weight of its volume of the fluid. The point of application of this force is the center of figure* of the submerged body and it is called the *center of buoyancy*. This principle was first enunciated by Archimedes, and it is called *Archimedes' principle*. It may be made almost self-evident by the following considerations: Given a fluid at rest. *Imagine a certain portion of this fluid of any size and shape.* This *portion* is stationary, and therefore the surrounding fluid pushes upwards upon it with a force which is equal to its weight, and the point of application of this upward force is the center of mass of the portion. But the surrounding fluid acts upon the given portion in exactly the same way that it would act upon a submerged body of the same size and shape.

A body which is partly submerged in a liquid is pushed upwards by a force which is equal to the weight of the displaced volume of liquid, and the point of the application of this upward force is the center of figure of the submerged part of the body. Therefore a floating body displaces its weight of the liquid in which it floats.

Examples. — The buoyant force of water is familiar in a general way to everyone. The principle of Archimedes is utilized in the ordinary method of finding the specific gravity of a body as follows: The body is weighed in the air and then it is suspended under water and weighed again. The difference is the weight (mass) of its volume of water, and the specific gravity of the body

* The center of mass of the body, if the body is homogeneous.

may then be calculated according to the principles enunciated in Art. 8.

When a body is weighed on a balance its weight (mass) is underestimated if it is more bulky than the weights that are used to balance it; this is on account of the greater buoyant force exerted by the air on the body than on the weights. This error is often quite appreciable, and it must be allowed for in accurate weighing.

The buoyant force of the air is most strikingly shown by the balloon.

116. Equilibrium of a floating body.* — A body is said to be in *unstable equilibrium* when the forces which act upon it tend to carry it farther and farther from its equilibrium position when it is displaced slightly therefrom. Thus a body standing vertically on a sharp point is in unstable equilibrium, the least displacement of the body in any direction causes it to fall over.

A body is said to be in *neutral equilibrium* when the forces which act upon the body remain in equilibrium as the body moves. Thus a homogeneous sphere resting on a smooth horizontal table, and a balanced wheel supported on an axle are in neutral equilibrium.

A body is said to be in *stable equilibrium* when the forces which act upon it tend to bring it back to its equilibrium position when it is displaced therefrom. Thus a weight fixed to the end of a spring, a pendulum hanging vertically downwards, and a block resting on a table are in stable equilibrium.

A body is said to have a high degree of stability when a very considerable force is required to displace it from its equilibrium position. Thus a broad sail-boat with its ballast placed low down in its hold is very stable, because a very considerable force is required to turn the boat from its vertical position.

Condition of equilibrium of a floating body. — When a floating body is stationary, it is, of course, in equilibrium and the downward force of gravity must have the same line of action as the

* This subject is treated in detail in works on naval architecture.

upward force of buoyancy, otherwise these two forces would have an unbalanced torque action and the body would not be in equilibrium. Therefore *the center of a mass of a floating body and the center of figure of the submerged portion of the body (center of buoyancy) must lie in the same vertical line.*

The problem of determining the degree of stability of a floating body is greatly complicated by the change of shape of the submerged part of the body when the body is tilted to one side, and the shifting of the center of buoyancy which is due to this

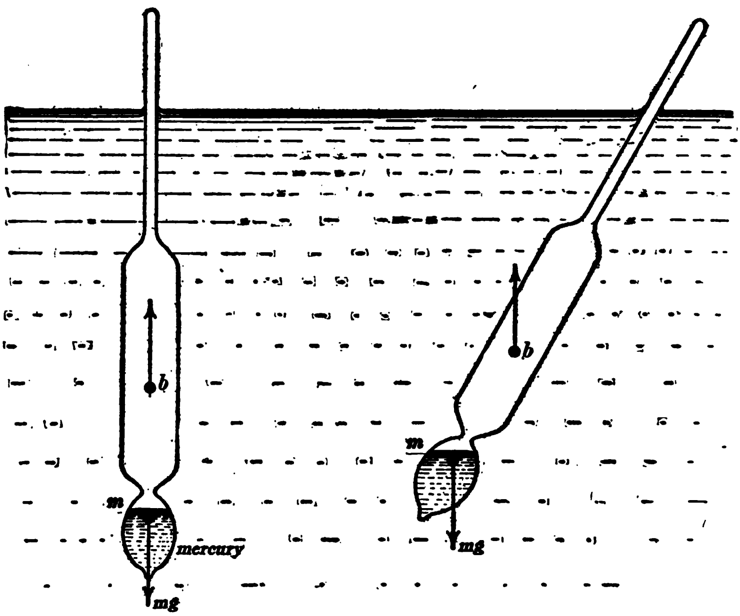


Fig. 118. Fig. 119.

change of shape. Therefore the simplest case is that of a floating body of which the submerged portion does not change its shape when the body is tilted to one side.

Examples of simplest case.—The submerged part of a floating sphere is the same in shape however the sphere be turned and therefore the center of buoyancy does not move as the sphere is turned. If the center of mass of the sphere is at its geometrical

center we have a case of neutral equilibrium of floating; if the sphere is heavier on one side, it floats in stable equilibrium with its heavy side downwards, and in unstable equilibrium with its heavy side upwards.

The most interesting simple example of equilibrium of floating

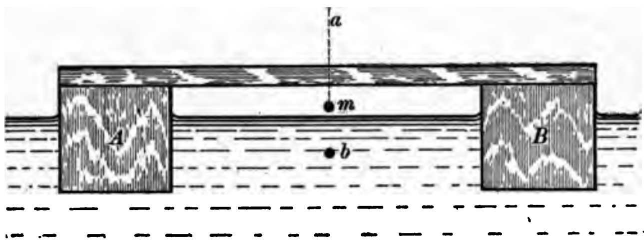


Fig. 120.

is the hydrometer as shown in Figs. 118 and 119. The shape of the submerged portion is slightly altered when the hydrometer is tipped over, but the change of shape is nearly negligible and the center of buoyancy b is nearly fixed in position.

Example of the general case.—Consider two floats A and B,

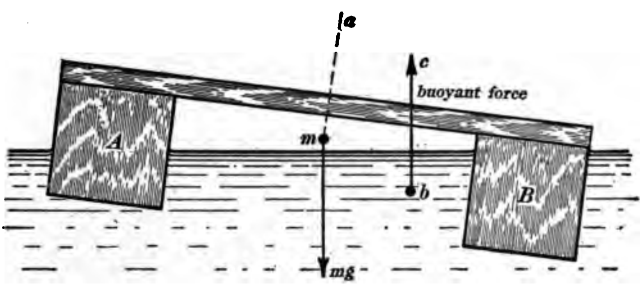


Fig. 121.

Figs. 120 and 121, connected rigidly together by a beam. This arrangement is similar to the style of boat called a catamaran, and when it is in equilibrium the center of mass m and the center of buoyancy b are located as shown in Fig. 120. When, however, the arrangement is tilted, as shown in Fig. 121, the center of buoyancy b shifts towards the lower side, while the center of

mass m of course remains stationary. *The arrangement behaves, for slight angles of tilting, as if its center of buoyancy were fixed in the line ma at the place B where the line ma is cut by the vertical line bc in Fig. 121, because the line of action of the buoyant force passes through the point B for any small angle of tilting.* The point B is called the *metacenter* of the float.

117. The hydrometer. — The common form of the hydrometer is a light glass float, weighted at one end with lead or mercury, and having a cylindrical glass stem at the other end as shown in Fig. 118. This float sinks to different depths in liquids of different specific gravities and upon the stem is a scale which indicates the specific gravity of the liquid in which the instrument is placed.

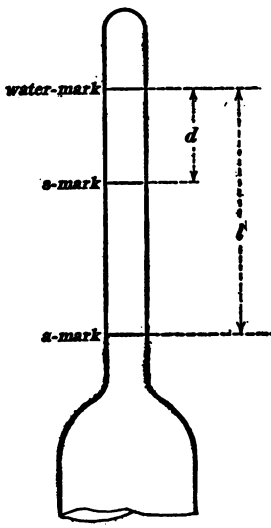


Fig. 122.

The specific gravity scale. — To construct a specific gravity scale on the stem of a hydrometer, the instrument is floated in water and the water-mark located, the instrument is then floated in a liquid of known specific gravity a, the a-mark is located, and then the distance l between the water-mark and the a-mark is measured. The scale is then determined by calculating the distance from the water-mark to each desired mark of the scale. Thus the distance d from the water-mark to the s-mark (s being a specific gravity, 1.10, 1.20, etc.) is given by the formula

$$d = l\left(\frac{1 - \frac{1}{s}}{1 - \frac{1}{a}}\right) \tag{62}$$

This equation may be derived as follows: A floating body displaces its weight of a liquid. The volume of water displaced by

the instrument being taken as unity, the volume below the a-mark is $1/a$ and the volume below the s-mark is $1/s$ inasmuch as these liquids are supposed to be a times as heavy and s times as heavy as water respectively. Therefore the volume of the length l of the stem is $(1 - 1/a)$ and the volume of the length d of the stem is $(1 - 1/s)$, and, the stem being assumed cylindrical, the lengths l and d are proportional to these volumes.

Beaumé hydrometer scales. — The specific gravity scale on a hydrometer is not a scale of equal parts and therefore the construction of the scale is tedious. One account of this fact a number of schemes have been proposed for constructing hydrometers with arbitrary scales of equal parts. Of these scales those of Beaumé are most extensively used.

Beaumé's scale for heavy liquids is constructed by locating the water-mark (near the top of the stem), and the mark to which the instrument sinks in a 15 per cent. solution of pure sodium chloride (common salt). The space between these marks is divided into 15 equal parts, and divisions of like size are continued down the stem. These divisions are numbered downwards from the water-mark. A liquid is said to have a specific gravity of 26° *Beaumé heavy* when the hydrometer sinks in it to mark number twenty-six on the scale here described.

Beaumé's scale for light liquids is constructed by locating the mark to which the instrument sinks in a 10 per cent. solution of sodium chloride (near the bottom of the stem), and the water-mark. The space between these marks is divided into 10 equal parts, and divisions of like size are continued up the stem. These divisions are numbered upwards from the salt solution mark. A liquid is said to have a specific gravity of 17° *Beaumé light* when the hydrometer sinks to mark number seventeen on the scale here described.

CAPILLARY PHENOMENA OF LIQUIDS.

118. Cohesion; adhesion. — When a body is under stress, as for example a stretched wire, the tendency of the stress is to tear

the contiguous parts of the body asunder. The forces which oppose this tendency and hold the contiguous parts of a body together are called the forces of *cohesion.* The forces which cause dissimilar substances to cling together are called the forces of *adhesion.* The discussion of the elastic properties of solids is a discussion of their properties of cohesion. The cohesion of water and the adhesion between water and glass are the forces which determine the curious behavior of water in a fine hair-like tube of glass, and the phenomena exhibited by liquids because of cohesion and adhesion are called *capillary phenomena* from the Latin word *capillaris* meaning a hair.

119. Surface tension. — On account of their cohesion, all liquids behave as if their free surfaces were stretched skins, that is, as if their free surfaces were under tension. Thus a drop of a liquid tends to assume a spherical shape on account of its surface tension. A mixture of water and alcohol may be made of the same density as olive oil, and a drop of olive oil suspended in such a mixture becomes perfectly spherical.

Many curious phenomena * are produced by the variation of the surface tension of a liquid with admixture of other liquids or with temperature. Thus a drop of kerosene spreads out in an ever widening layer on a clean water surface, on account of the fact that the tension of the clean water surface beyond the layer of oil is greater than the tension of the oily surface. A small shaving of camphor gum darts about in a very striking way upon a clean water surface, on account of the fact that the camphor dissolves in the water more rapidly where the shaving happens to have a sharp projecting point, the water surface has a lessened tension where the camphor dissolves, and the greater tension on the opposite side pulls the shaving along. A thin layer of water on a horizontal glass plate draws itself away and leaves a dry spot where a drop of alcohol is let fall on the plate. A thin layer of lard on the bottom of a frying pan pulls itself away from

* See the very interesting article *capillary action* in the Encyclopedia Britannica. This article also gives a comprehensive discussion of the theory of capillary action.

the hotter parts of the pan and heaps itself up on the cooler parts, because of the greater surface tension of the cooler lard.

120. Angles of contact. Capillary elevation and depression. — The clean surface of a liquid always meets the clean walls of a containing vessel at a definite angle. Thus a clean surface of water turns upwards and meets a clean glass wall tangentially, and a clean surface of mercury turns downwards and meets a clean glass wall at an angle of $51° 8'$.

Since a clean water surface turns upwards and meets a glass wall tangentially it is evident that the surface of water in a small glass tube must be concave as shown in Fig. 123, and the result

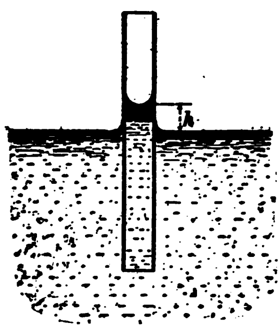


Fig. 123.

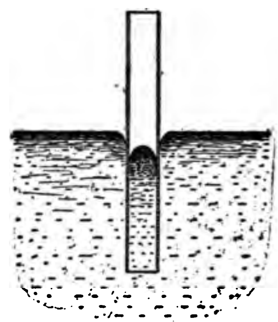


Fig. 124.

is that the water is drawn up into the tube. On the other hand, the surface of mercury in a small glass tube is convex and the surface tension pulls the mercury down below the level of the surrounding mercury as shown in Fig. 124.

121. Measurement of surface tension of water. — Let r be the radius of the bore of the glass tube in Fig. 123. Then the circumference $2\pi r$ is the width of the surface film of water at the point of tangency, and $2\pi rT$ is the total upward force due to the tension of the film, T being the tension per unit width. The volume of water in the tube above the level of the surrounding water is πr^2h, and the weight of this water is πr^2hdg where d is the density in grams per cubic centimeter and g is the accelera-

tion of gravity. The weight of water in the tube being supported by the tension of the film, we have

$$2\pi r T = \pi r^2 h d g\,,$$

whence

$$T = \frac{rhdg}{2}$$

from which T may be calculated when r, d, and g are known and h observed. The surface tension of water is found in this way to be 81 dynes per centimeter breadth.

PROBLEMS.

136. Calculate the number of dynes per square centimeter in one pound-weight per square inch, taking the acceleration of gravity equal to 980 cm./sec^2.

137. Figure 137p represents a hydrostatic press. The dis-

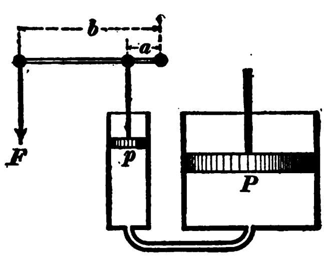


Fig. 137p.

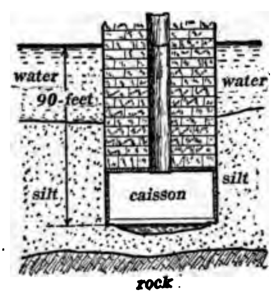


Fig. 140p.

tances a and b are equal to 6 inches and 6 feet respectively, the diameter of the pump plunger p is 1.5 inches and the diameter of the press plunger P is 24 inches. Find the total force on P due to a force of 100 pounds at F, neglecting friction.

138. Calculate the circumferential tension in the cylindrical shell of a boiler due to a steam pressure of 125 pounds per square inch, the diameter of the boiler being 6 feet.

139. Sheet steel 0.02 inch thick will safely stand a tension of 200 pounds per inch of width. What is the greatest diameter

of steel tube with 0.02 inch wall, which can safely withstand a pressure of 150 pounds per square inch?

140. Calculate the pressure of the air in the caisson shown in Fig. 140*p*; the distance from the water level in the river to the water level in the caisson being 90 feet.

141. Oil and water are drawn up in two connecting tubes as shown in Fig. 141*p*. The height of the water column is 36

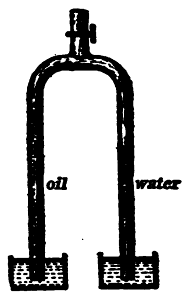


Fig. 141*p*.

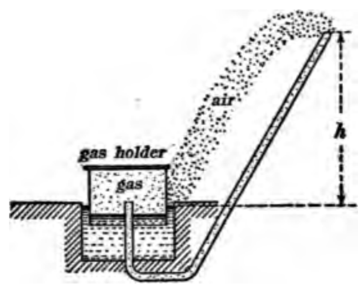


Fig. 142*p*.

inches, and the height of the oil column is 42 inches. What is the ratio of the densities of oil and water (specific gravity of the oil)?

142. The density of air under ordinary conditions is 0.0012 grams per cubic centimeter, and the density of illuminating gas is, say, 0.0008 grams per cubic centimeter. The pressure of illuminating gas at the base of the gas holder exceeds the pressure of the outside air at the same level by 2 inches of water. Find the difference between the gas and air pressures on top of a hill at a height h of 400 feet above the gas holder.

Note. — In this problem assume that the density of the gas is uniform and that the density of the air is uniform. The density of water is one gram per cubic centimeter and one inch equals 2.54 centimeters.

143. Sketch the form of a dish such that the total force due to hydrostatic pressure on its bottom shall be (*a*) greater than, (*b*) equal to, and (*c*) less than the weight of the contained liquid.

144*a*. The density of mercury at 0° C. is 13.5956 grams per cubic centimeter. Calculate the value in dynes per square centimeter of standard atmospheric pressure, namely 76 cm. of mer-

cury at 0° C., the value of gravity being 980.61 cm. per second per second. Give the result in dynes per square centimeter.

144*b*. The specific gravity of mercury is approximately 13.6. The pressure in pounds per square inch at a point x feet beneath pure water is $p = 0.434x$. Find the value in pounds per square inch of one English standard atmosphere, namely, 30 inches of mercury.

145. Calculate the height of the *homogeneous atmosphere;* that is, assuming that the atmosphere has a uniform density of 0.00129 grams per cubic centimeter throughout, calculate the depth which would produce standard atmospheric pressure.

146. A masonry dam 12 feet high is to be built on a flat rock bottom. The dam is to have the shape of a rectangular parallelopiped (in order that this problem may be simple). How thick must the dam be made so that the force of the water will just be unable to tip it over? The masonry weighs 120 pounds per cubic foot.

Note. — The moment of the force with which the water pushes on the dam is to be equal to the moment of the gravity-pull on the dam, both moments being taken about the down-stream, bottom edge of the dam.

147. A water gate 36 inches wide has its upper edge 2 feet beneath the level of the water in a canal lock, and its lower edge 5 feet beneath the water level. What is the total force on the gate and what is the distance, beneath the water level, of the point of application of this total or resultant force?

148. A piece of lead weighs 233.60 grams in air and 212.9 grams in water at 20° C. What is the specific gravity and the density of lead at 20° C.?

For table of densities of water at various temperatures see chapters on heat.

149. A piece of glass weighs 260.7 grams in air and 153.8 grams in water at 20° C. The same piece of glass weighs 92.2 grams in H_2SO_4 at 20° C. What is the specific gravity of H_2SO_4 at 20° C.?

150. A glass bulb weighs 75.405 grams when filled with air at standard temperature and pressure. It weighs 74.309 grams

when the air is pumped out. It weighs 74.422 grams when filled with hydrogen at the same temperature and pressure. What is the specific gravity of hydrogen referred to air?

151. What is the net lifting capacity of a balloon containing 400 cubic meters of hydrogen, its material weighing 250 kilograms? (Weight of a cubic meter of air is 1,200 grams; weight of a cubic meter of hydrogen is 90 grams.)

152. The distance along a hydrometer stem from the water mark to the mark to which the instrument sinks in kerosene (specific gravity 0.79) is 9.62 centimeters. Calculate the distance from the water mark to the marks to which the instrument would sink in a 20% solution of alcohol, in a 40% solution of alcohol, in a 60% solution of alcohol, in an 80% solution of alcohol, and in pure alcohol. The specific gravities of these solutions are as follows: 20% = 0.975; 40% = 0.951; 60% = 0.913; 80% = 0.863; 100% = 0.794.

153. The specific gravity of a 15 per cent. solution of sodium chloride at ordinary room temperature is 1.1115. Calculate the specific gravity corresponding to 26° Beaumé (heavy).

Note. — This problem is to be solved by using equation (62) in a way that will be apparent when it is considered that degrees Beaumé represent distances along the hydrometer stem.

154. The specific gravity of a 10 per cent. solution of sodium chloride at ordinary room temperature is 1.0734. Calculate the specific gravity corresponding to 20° Beaumé (light).

155. Two rectangular boxes 12 inches × 12 inches × 16 feet are fastened to two cross-beams like a catamaran, and the whole weighs 625 pounds. The distance apart from center to center of the two boxes is 6 feet. Find how far to one side the center of buoyancy shifts when the raft is tilted 5° about its longitudinal axis, find the approximate position of the metacenter, and find the torque tending to bring the raft into a horizontal position.

Note. — For the sake of simplicity assume that the submerged part of each box remains rectangular. The metacenter is defined in terms of an infinitesimal angle of tilting, but its position for a 5° tilt may be determined without the use of calculus in the case here considered.

CHAPTER X.

HYDRAULICS.

[Throughout this chapter the following units are used: feet, feet per second, square feet, cubic feet, pounds (mass), pounds per cubic foot (density), pounds (force), pounds per square foot (pressure), and foot-pounds (energy). The factor g is equal to 32.2 which is the acceleration in feet per second per second of a one pound body when acted upon by an unbalanced force of one pound-weight.]

122. Limitations of this chapter. — Hydraulics, in the general sense in which the term is here used, is the study of liquids and gases in motion; and the phenomena which are presented in this branch of physics are excessively complicated. Even the apparently steady flow of a great river through a smooth sandy channel is an endlessly intricate combination of boiling and whirling motion; and the jet of spray from a hydrant, or the burst of steam from the safety-valve of a locomotive, what is to be said of such things as these? Or let one consider the fitful motion of the wind as indicated by the swaying of trees and as actually visible in driven clouds of dust and smoke, or the sweep of the flames in a conflagration! These are *actual* examples of fluids in motion, and they are indescribably, infinitely * complicated. The finer details of such phenomena, however, are devoid of

* Everyone concedes the idea of infinity which is based upon abstract numerals (one, two, three, four and so on *ad infinitum*!), and the idea of infinity which is based on the notion of a straight line; but most men are wholly concerned with the humanly significant and persistent phases of the material world, their perception does penetrate into the substratum of utterly confused and erratic action which underlies every physical phenomenon, and they balk at the suggestion that the phenomena of fluid motion, for example, are infinitely complicated. Surely the abstract idea of infinity is as nothing compared with the awful intimation of infinity that comes from things that are seen and felt.

practical significance, indeed they present but little that is sufficiently definite even to be intelligible.

The science of hydraulics is based on ideas which refer to general aspects of fluid motion, like a sailor's idea of a ten-knot wind; and, indeed, the engineer is concerned chiefly with what may be called *average effects* such as the time required to draw a pail of water from a hydrant, the loss of pressure in a line of pipe between a pump and a fire nozzle, or the force exerted by a water jet on the buckets of a water wheel. These are called average effects because they are never perfectly steady but always subject to perceptible fluctuations of an erratic character, and to think of any of these effects as having a definite value is of course to think of its average value under the given conditions.* The extent to which the practical science of hydraulics is limited is evident from the following outline of the ideal types of flow upon which nearly the whole of the science is based.

Permanent and varying states of flow. — When a hydrant is suddenly opened, it takes an appreciable time for the flow of water to become steady. During this time (*a*) *the velocity at each point of the stream is increasing and perhaps changing in direction also.* After a short time, however, the flow becomes fully established and then (*b*) *the velocity at each point in the stream remains unchanged in magnitude and direction.*† The motion (*a*) is called a *varying state of flow*, and the motion (*b*) is called a *permanent state of flow.* Most of the following discussion applies to permanent states of flow, indeed there are but few cases in which it is important to consider varying states of flow.

The idea of simple flow. Stream lines. — The idea of simple flow applies both to permanent and to varying states of flow, but it is sufficient to explain the idea in its application to permanent flow only. When water flows steadily through a pipe, the motion

* See two brief articles by W. S. Franklin, *Transactions of American Institute of Electrical Engineers*, Vol. XX, pages 285–286; and *Science*, Vol. XIV, pages 496–497, September 27, 1901.

† Assuming the stream to be free from turbulence. See the following definition of simple flow.

is always more or less complicated by continually changing eddies, the water at a given point *does not* continue to move in a fixed direction at a constant velocity; nevertheless, it is convenient to treat the motion as if the velocity of the water were in a fixed direction and constant in magnitude at each point. Such a motion is called a *simple flow*. In the case of a simple flow a line

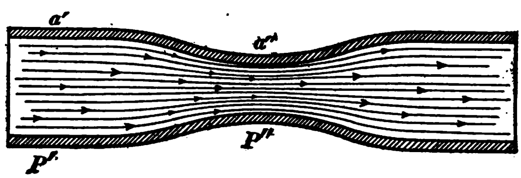


Fig. 125.

can be imagined to be drawn through the fluid so as to be at each point in the direction of the flow at that point. Such a line is called a *stream line*. Thus the fine lines in Fig. 125 are stream lines representing a simple flow of water through a contracted part of a pipe. To apply the idea of simple flow to an actual case of fluid motion is the same thing as to consider

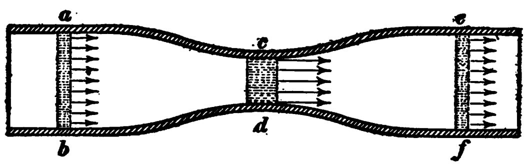


Fig. 126.

the *average character of the motion during a fairly long interval of time.*

Lamellar flow.—Even though the motion of water in a pipe may be approximately a simple flow, the velocity may not be the same at every point in a given cross-section of the pipe, that is, the velocity may not be the same at every part of the layer *ab*, Fig. 126; in fact the water near the walls always moves slower than the water near the center of a pipe; nevertheless, it is convenient in many cases to treat the motion as if the velocity were the same at every point in any layer like *ab*, Fig. 126. Such an

ideal flow is called a *lamellar flow*, because in such a flow the fluid in any layer or lamella *ab* would later be found in the layer *cd*, and still later in the layer *ef*. To apply the idea of lamellar flow to an actual case of fluid motion is the same thing as to consider the *average velocity over the entire cross-section of a stream.*

Rotational and irrotational flow. — In certain cases of fluid motion each particle of the fluid, if suddenly solidified, would be found to be rotating at a definite angular velocity about a definite axis; such fluid motion is called *rotational motion* or *vortex motion*. Thus the whirling motion of the water in an emptying sink is vortex motion. In other cases of fluid motion the particles of the fluid are not rotating; this kind of fluid motion is called *irrotational motion*. Some of the important practical aspects of vortex motion are discussed in the Encyclopedia Britannica article *Hydromechanics, Part III., Hydraulics*, sections 30, 31, 103 and 190.

In irrotational fluid motion the velocity can be represented as a potential gradient, whereas in rotational fluid motion the velocity cannot be represented as a potential gradient. See Art. 21, space variation of vectors.

123. Some actual phenomena of fluid motion. — The following treatment of fluid motion is so largely based upon the idea of simple lamellar flow that in pursuing the discussion we will be carried far away from any consideration of the details of actual

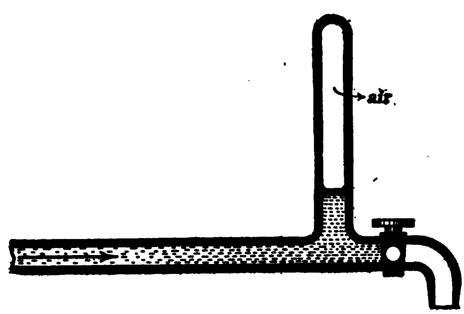


Fig. 127.

fluid motion, and, although many of these details are essentially erratic, still there are a few details which are definitely typical.

The water hammer. — The most striking phenomenon that is associated with a varying state of fluid motion is the effect produced when an open hydrant is suddenly closed; the momentum of the water in the pipe causes the water to exert on the

suddenly closed valve a momentary force very much like a hammer blow. This momentary force is often excessively large in value and a valve which is closed suddenly should be protected by an air cushion as shown in Fig. 127. The sharp rattling noise which is occasionally produced in steam pipes is due to the "water hammer." A column of condensed water is driven along the pipe by the steam, the cooler steam ahead of the column condenses, and the column of water hammers against the end of the pipe or against a stationary body of water in the pipe.

The hydraulic ram consists of a valve A, Fig. 128, arranged to automatically open and close the end of a long pipe PP.

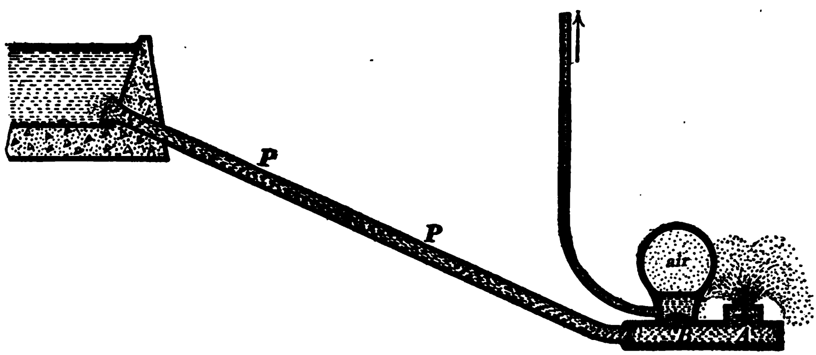


Fig. 128.

When the valve opens the water from the dam starts to flow, this flow lifts the valve A which suddenly closes the end of the pipe PP, and the momentum of the water in PP generates a momentary pressure which lifts the valve B and forces a small quantity of water to a high storage tank. The valve A then falls, and the action is repeated. The automatic opening of the valve A is due to the recoil of the water in the pipe PP as follows: At the moment when the water in PP is brought to rest in forcing water into the storage tank, the pressure at the end of PP is of course still excessive and the water near the end of PP is compressed. This compression then relieves itself by starting a momentary backward flow, or recoil, of the water in PP,

and this recoil is followed by a momentary decrease of pressure sufficient to allow the valve A to drop.

The sensitive flame. — When a fluid flows through a fairly smooth walled pipe, the motion approximates very closely to a simple flow if the velocity is not excessive; but when the velocity is increased the motion tends to become more and more turbulent (full of eddies), and in many cases there is a fairly definite velocity at which the motion suddenly becomes very turbulent. This is shown by watching the movement of "sawdust-water" through a large glass tube. At low velocities the particles of sawdust move in fairly straight paths, but as the velocity of flow is increased the particles begin to gyrate with considerable violence when a certain velocity is reached.

This sudden increase of turbulence is illustrated by the familiar behavior of a gas flame. When the gas is turned on more and more the flame remains fairly steady until the velocity of the flowing gas reaches a certain critical value and then the flame suddenly becomes rough and unsteady. When the flame is on the verge of becoming unsteady it is sometimes very sensitive; the least hissing noise causes it to become turbulent. An extremely sensitive flame may be obtained by burning ordinary illuminating gas from a smooth circular nozzle made by drawing a glass tube down to the desired size (about $\frac{1}{2}$ millimeter to 1 millimeter diameter of opening). Generally, several nozzles must be tried before one is found that is suited to the gas pressure that is available.

Vortex rings. — When a fluid is at rest, mixing takes place only by the very slow process of diffusion,* and when a fluid is in turbulent motion the mixing of the different parts of the fluid takes place very rapidly on account of the eddies which constitute the turbulence. The slowness of mixing of a smoothly flowing fluid, however, is illustrated by the smooth gas flame and by the threads of smoke that rise from the end of a cigar. Such a stream of fluid flowing smoothly through a large body of fluid at

* See chapters on heat.

rest tends always to break up into what are called vortex rings. Thus a fine jet of colored water entering at the top of a large vessel of clear water and streaming towards the bottom, breaks up into rings which spread out wider and wider as they move downwards, each ring preserving its identity (not mixing with the clear water). The most interesting example of the formation of vortex rings is the familiar case of the formation of smoke rings when smoke issues as a moderate puff from an orifice into the

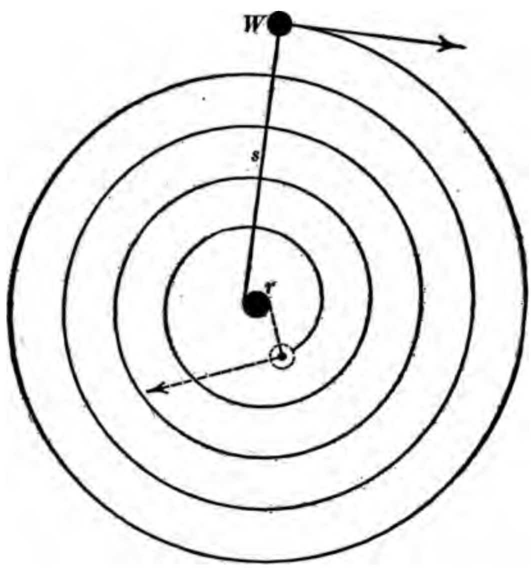


Fig. 129.

air. Of course the smoke only serves to make the rings visible, and a candle can be blown out by invisible vortex rings projected across a large room from an orifice in a box by striking a flexible diaphragm which is stretched like a drum head across the open side of the box.

Cyclonic movements.—When water flows out of a hole in the bottom of a sink a whirlpool generally forms above the hole. In order to explain this rapid whirling motion let us consider a weight W, Fig. 129, which is twirled on a string that winds up

on a rod r so as to bring the weight continually nearer to the center. The velocity of the weight tends to remain unchanged in value and therefore the number of revolutions per second tends to become greater and greater as the weight approaches the center.* The formation of a whirlpool in an emptying sink is due to a chance rotating or whirling motion of the water in the sink which may be imperceptible, but which becomes very greatly exaggerated as the water flows towards the hole from all sides.

The rotation of the earth on its axis involves a slow motion of turning of one's horizon about a vertical axis (except at the equator). When the warm air near the earth's surface starts to flow upwards at a given point, a chimney-like effect is produced by the rising column of warm air, the lower layers of warm air flow towards this "chimney" from all sides, and the slow turning motion of the horizon becomes very greatly exaggerated in a more or less violent whirl at the "chimney" which is the center of the storm. The *cyclone* is a storm movement of this kind covering hundreds of thousands of square miles of territory with a central chimney hundreds of miles in diameter, the *tornado* is a storm movement of this kind covering only a few square miles of territory with a central chimney seldom more than a thousand yards in diameter. The whirling motion near the center of a tornado is often excessively violent.

124. Rate of discharge of a stream. — The volume of water which is delivered per second by a stream is called the *discharge rate* of the stream. Thus the mean discharge rate of the Niagara river is 300,000 cubic feet per second. The rate of discharge of a stream is equal to the product of the average velocity, v, of the stream and its sectional area a. For example, let PP, Fig. 130, be the end of a pipe out of which water is flowing, and let us assume that the velocity of flow has the same value v over the entire section of the stream (lamellar flow), then the water which flows out in t seconds would make a prism of which the length

* Let the student hang by his hands from the end of a long rope with his feet held as far apart as possible, let a comrade set him spinning, and then let the student bring his feet together.

is vt and the end area is a. That is the volume avt of water flows out of the pipe in t seconds so that the discharge rate is av.

Variation of velocity with sectional area of a steady stream. — Consider a simple flow of water through a pipe as indicated by the stream lines in Fig. 125. Let a' and a'' be the cross-sectional areas of the stream at any two points P' and P'', and let v' and

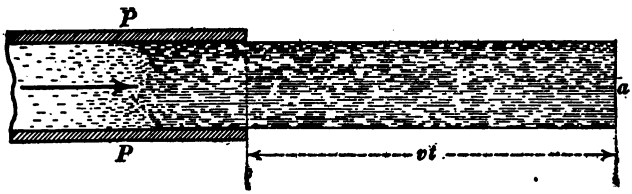


Fig. 130.

v'' be the average velocities of the stream at P' and P'' respectively. Then $a'v'$ is the volume of water which passes the point P' per second and $a''v''$ is the volume of water which passes the point P'' per second; and, therefore, since the same amount of water must pass each point per second, we have

$$a'v' = a''v'' \tag{63}$$

that is, the product av has the same value all along the pipe, so that v is large where a is small, and v is small where a is large.

Equation (63) applies to a fluid which is approximately incompressible; that is, to a liquid, a given amount of liquid having always sensibly the same volume so that the volume $a'v'$ which passes one point of a steady stream per second is equal to the volume $a''v''$ which passes any other point of the same stream per second. In the case of a gas, however, the volume changes very perceptibly with pressure and equation (63) becomes

$$a'v'd' = a''v''d'' \tag{64}$$

where a' is the sectional area of the steady gas stream at one place, v' is the average velocity of the stream at that place, d' is the density of the gas at that place, and a'', v'' and d'' are the

cross-sectional area and velocity of the stream and the density of the gas at another part of the stream.

125. The ideal frictionless incompressible fluid. — When a jet of water issues from a tank, there is a certain relation between the velocity of the jet and the difference in pressure inside and outside of the tank. When there are variations of the velocity of flow of water through a pipe due to enlargements or contractions of the pipe [see equation (63)], the pressure decreases wherever the velocity increases and *vice versa.* These mutually dependent changes of velocity and pressure are always complicated by friction, and by the variations of the density of the fluid due to the variations of pressure; and in order to gain the simplest possible idea of these mutually dependent changes of velocity and pressure the conception of the *frictionless incompressible fluid* is very useful.

When the water in a pail is set in motion by stirring, it soon comes to rest when it is left to itself. A fluid which would continue to move indefinitely after stirring would be called a frictionless fluid.

When a moving fluid is brought to rest by friction, the kinetic energy of the moving fluid is converted into heat and lost. Such a loss of energy would not take place in a frictionless fluid, and therefore the total energy (kinetic energy plus potential energy) of a frictionless fluid would be constant. *This principle of the constancy of total energy is the basis of the following discussion of the flow of the ideal frictionless fluid.* The following discussion applies to fluids which are also incompressible. In fact, ordinary liquids are nearly incompressible. The flow of gases is discussed in the chapters on heat.

126. Energy of a liquid. (*a*) *Potential energy per unit of volume.* — Work must be done to pump a liquid into a region under pressure, the amount of work done in pumping one unit of volume of the liquid is the potential energy per unit of volume of the liquid in the high pressure region, and it is equal to the pressure. That is

$$W' = p \tag{65}$$

In this equation W' and p may both be expressed in c.g.s. units or in the units enumerated at the head of this chapter.

Proof of equation (65).—Let CC, Fig. 131, be the cylinder of a pump which is used to pump liquid into a tank under a pressure of p pounds per square foot, and let the area of the piston be a square feet. Then the force required to move the piston (ignoring friction) is ap pounds, and the work done in moving the piston through a distance of l feet is apl foot-pounds. But al is the volume of water pushed into the tank

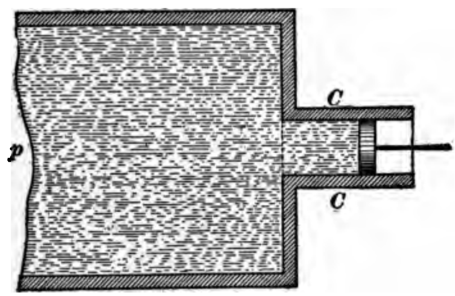


Fig. 131.

by the movement of the piston, and therefore, dividing apl by al gives the work in foot-pounds required to push one cubic foot of water into the tank.

When a stream of liquid moves in a horizontal plane, the gravity pull of the earth does no work on the liquid; but when a stream flows through an inclined pipe the gravity pull of the earth does work (positively or negatively) on the liquid and it is necessary therefore in this case to consider the energy of altitude as a part of the potential energy of the liquid. In fact one cubic foot of liquid of which the weight is d pounds has a potential energy equal to hd foot-pounds when it is at a height of h feet above a chosen reference level, so that the total potential energy of the liquid per cubic foot is

$$W' = p + hd \tag{66}$$

(*b*) *Kinetic energy.* — Let v be the velocity in feet per second of a moving liquid, and let d be the mass of one cubic foot of the liquid in pounds (d is the density of the liquid). Then the kinetic energy of one cubic foot of the liquid in foot-pounds is

$$W'' = \frac{1}{2g} dv^2 \tag{67}$$

according to equation (27) of chapter VI.

127. Efflux of a liquid from a tank. — Consider a tank, Fig. 132, containing a liquid of which the density is d pounds per cubic foot. Let oo be an orifice from which the liquid issues as a jet at a velocity v feet per second to be determined. Let p pounds per square foot be the pressure in the tank at the level of the orifice, and let p' be the outside pressure (atmospheric pressure). In the tank, where the velocity of the liquid is inappreciable, the total energy of the liquid per unit of volume is the potential energy p [equation (65)]. In the jet the total energy per unit volume is $p' + 1/2g \times dv^2$ [equations (65) and (67)]. As a portion of the liquid moves from the tank into the jet its total energy would remain unchanged if it were frictionless so that we would have

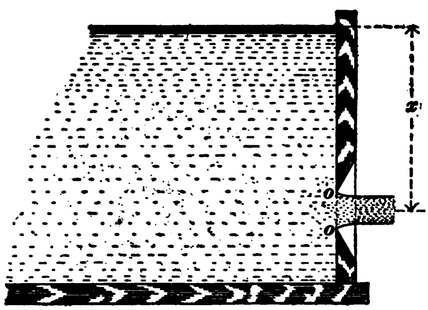


Fig. 132.

$$p = p' + \frac{1}{2g} dv^2$$

whence

$$v = \sqrt{\frac{2g(p-p')}{d}} \tag{68}$$

This equation expresses the velocity of efflux of a frictionless incompressible fluid. The effect of friction is to decrease v, and the effect of compressibility is to increase v. For ordinary liquids

the effect of friction is the greater, and equation (68) gives too large a value for v. For gases the effect of compressibility is the greater, and equation (68) gives too small a value for v.

Torricelli's theorem. — The velocity of efflux of a frictionless liquid is equal to the velocity a body would gain in falling freely through the distance x of Fig. 132. The pressure-difference $p - p'$ is equal to xd, according to equation (60)* of chapter IX, so that, substituting xd for $p - p'$ in equation (68), we have

$$v = \sqrt{2gx}$$

and this is the velocity gained by a body in falling through the distance x, according to Art. 35.

128. Diminution of pressure in a throat. — A contracted portion of a pipe is called a *throat*. When a fluid flows through a

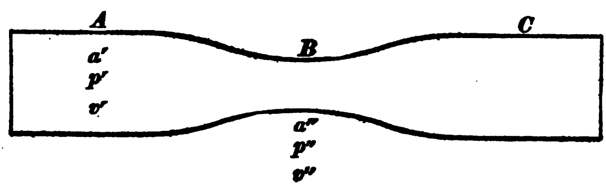


Fig. 133.

pipe in which there is a throat *the velocity of the fluid in the throat is greater* than it is in the larger portions of the pipe, and therefore *the pressure of the fluid in the throat is less* than it is in the larger portions of the pipe. Let Fig. 133 represent a pipe with a throat; let a' be the cross-sectional area of the pipe at A, and let p' and v' be the pressure and velocity respectively of the fluid at A; and let a'', p'' and v'' be the corresponding quantities at B. Then, if the fluid is incompressible, we have

$$a'v' = a''v'' \tag{i}$$

according to equation (63), and if the fluid is also frictionless we have

* It is to be remembered that in equation (60) c.g.s. units are used. This equation becomes $p = xd$ for the units used in this chapter.

$$p' + \frac{1}{2g}dv'^2 = p'' + \frac{1}{2g}dv''^2 \tag{ii}$$

Therefore, substituting the value of v'' from (i) in (ii) and solving for $p' - p''$, we have

$$p' - p'' = \frac{1}{2g}\left(\frac{a'^2 - a''^2}{a''^2}\right)dv'^2 \tag{69}$$

where $p' - p''$ is the diminution of pressure in the throat.

The diminution of pressure in a throat is explained directly from Newton's second law of motion as follows: Consider a particle of liquid at A, Fig. 133. This particle gains velocity as it approaches B, and loses velocity again as it approaches C. Therefore an unbalanced force must be pushing the particle forwards as it passes from A to B, that is, the pressure behind the particle is greater than the pressure ahead of the particle; and an unbalanced force must be opposing the motion of the particle as it passes from B to C, that is, the pressure ahead of the particle is greater than the pressure behind it.

Examples. — The diminution of pressure as a stream contracts into a throat and the rise of pressure as the stream widens out again are illustrated by several familiar devices as follows:

(*a*) *The disk paradox.* — Figure 134 represents a short piece of tube T ending in a flat disk DD, and dd is a light metal disk which is prevented from moving sidewise by a pin which projects into the end of the tube T. If one blows hard through the tube T the disk dd is held tight against DD because of the low pressure in the very greatly contracted portion of the air stream between the disks. In fact, the pressure of the air in the region between the disks is less than atmospheric pressure, and it increases towards the edge of the disks as the velocity of the air stream diminishes (and the sectional area of the stream increases).

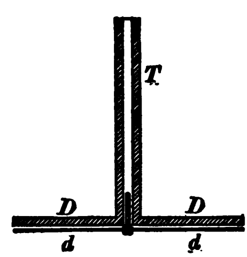


Fig. 134.

(*b*) *The jet pump.* — The essential features of the jet pump are shown in Fig. 135. Water from a fairly high pressure supply H enters a narrow throat, the low pressure in the throat sucks water from AA, and the water from H, together with the water from AA, is discharged into the reservoir R. This type of pump is frequently used for pumping water out of cellars, and it is extensively used as an air pump in chemical laboratories.

The steam injector is a jet pump, and its paradoxical action in pumping water into a boiler at the same pressure as the steam sup-

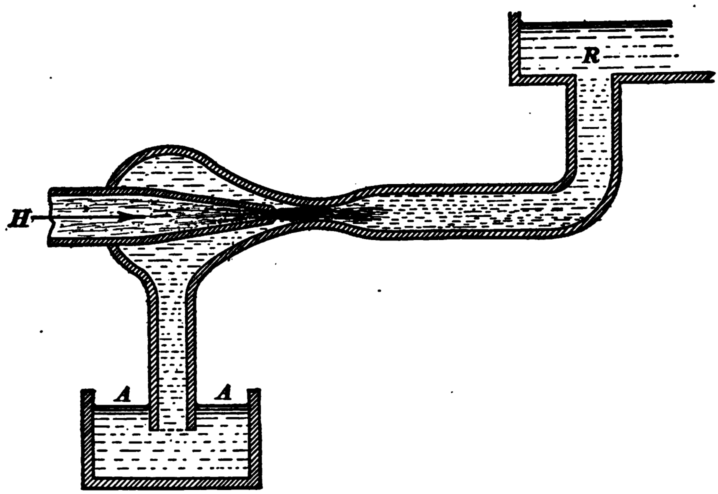


Fig. 135.

ply (or even higher), depends upon the low density of the steam. It is evident from equation (68) that the low density steam must acquire a very high velocity in flowing out of the boiler, whereas a very much lower velocity suffices to carry the water (including the condensed steam) back into the boiler.

(*c*) The volume of water discharged per second from a given sized orifice oo, Fig. 136, is greatly increased by the flaring tube AB. The rate of discharge of a frictionless fluid would depend only upon the size of the open end B of the tube, the contraction at A would have no effect. In the case of an actual fluid, the

effect of the contraction at A is to increase the friction considerably and thus reduce the discharge rate below what it would be if the tube at A were as large as at B.

(*d*) *The Venturi water meter* consists of a throat inserted in a water pipe through which the water to be measured flows. The

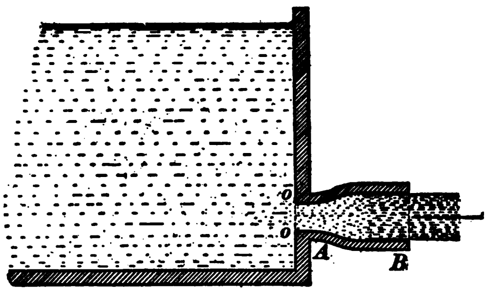


Fig. 136.

diminution of pressure $p'-p''$ [see equation (69)] is measured, and, since the cross-sectional areas a' and a'' are known, the velocity v' and the rate of discharge $a'v'$ can be calculated from the measured value of $p'-p''$.

129. Reaction of a water jet. Force of impact of a jet.—Figure. 137 represents a tank containing water at pressure p (in excess of outside pressure). The tank has an orifice of area a and the orifice is closed by a plug P. The force acting on the plug is equal to pa, and the total force pushing on the side AA of the tank is equal to total force pushing on the side BB *including the force acting on the plug*. Therefore, it would seem that an unbalanced force equal to pa would push the tank

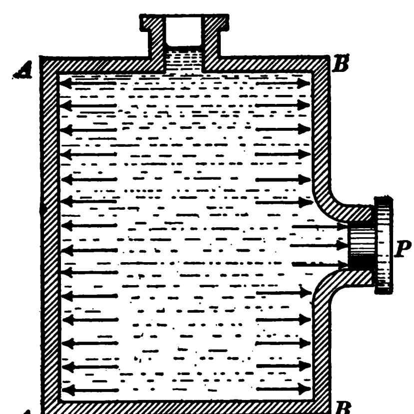


Fig. 137.

towards the left in Fig. 137 if the plug were removed; but when the plug is removed there is a reduction of pressure in the neighborhood of the orifice as indicated in Fig. 138, so that the unbalanced force which pushes the tank towards the left in Fig. 138 is much greater than pa, it is in fact equal to $2pa$ on the following assumptions, namely, (a) that the velocity of efflux is that of an ideal incompressible fluid, and (b) that the jet issues as a parallel stream of the same size as the orifice.

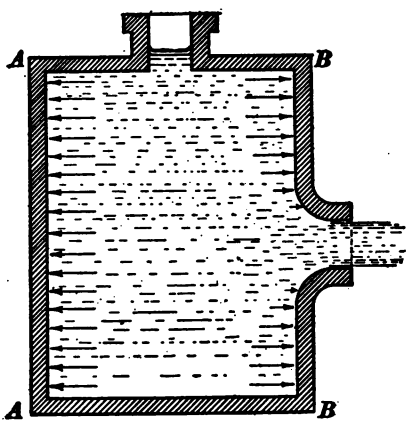


Fig. 138.

It is impossible, however, to show that the reaction of the jet is $2pa$ by considering the change of pressure inside of the tank due to the existence of the jet, but the reaction can be evaluated in a comparatively simple manner by considering the force which must act on the outflowing water to set it in motion. In one second av cubic feet or avd pounds of water flow out of the orifice, and this amount of water has gained velocity v. To impart velocity v to avd pounds in one second requires a force equal to $adv \times v \div g$ pounds-weight, according to equation (5), and of course the jet must push backwards upon the tank with an equal force. Therefore the reaction of the jet is adv^2/g pounds-weight; but the velocity of efflux and difference of pressure $p[=p'-p''$ of equation (68)] satisfy the equation

$$\frac{1}{2g}dv^2 = p$$

so that

$$\frac{1}{g}adv^2 = 2pa.$$

When a jet of water strikes an obstacle so as to be brought to rest, it exerts a force equal to adv^2/g on the obstacle. If the jet strikes a flat plate so as to rebound in a direction at right angles to its original velocity, as indicated in Fig. 139, then it

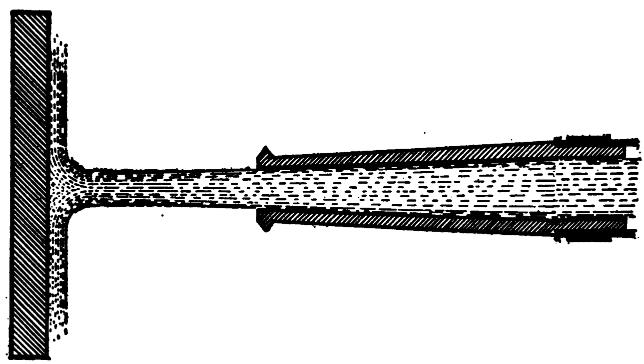


Fig. 139.

exerts the same force as it would exert if it were brought to rest, because it loses all of its velocity in the original direction. If the jet strikes a curved plate as indicated in Fig. 140 so as to rebound

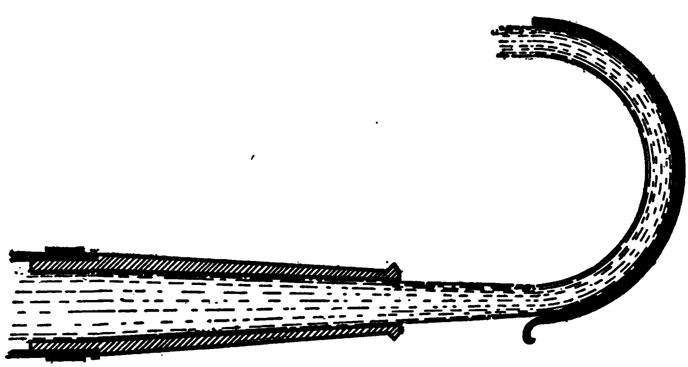


Fig. 140.

in an opposite direction with unchanged velocity (gliding along the curved plate without friction), then it would exert twice as much force as it would exert if it were brought to rest, because it

would lose its original velocity and gain an equal amount in the opposite direction.

The Pitot tube. — A glass tube drawn to a moderately fine point is placed in a stream of water moving at velocity v, as shown in Fig. 141. In accordance with what is stated above concerning the reaction and impact of a jet, the water in the tube should stand above the level of the stream at a height h which is approximately twice as great as the height which would cause an efflux velocity equal to v. That is, the velocity of the stream is approximately equal to $\sqrt{gh}$, according to Art. 127. This

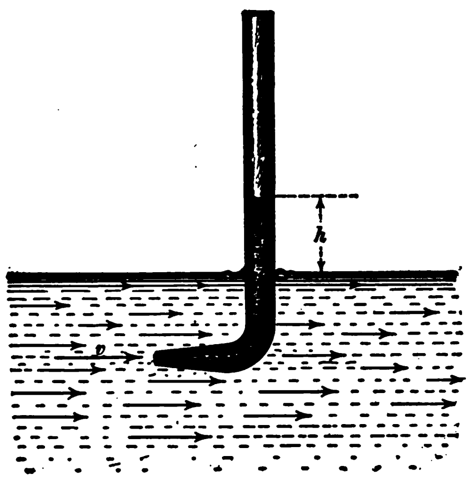


Fig. 141.

device is called the Pitot tube, it is frequently used for measuring the velocity of streams,* and, when so used, it is usually arranged as shown in Fig. 142, so as to bring the difference of level h into a convenient position for measurement. The tube A, Fig. 142, has its point directed against the stream, and the tube B has its point directed at right angles to the stream. By drawing the device, Fig. 142, through still water at a known velocity, or by

* Other methods for measuring the velocity of a stream are often used in practice. See, for example, Merriman's Hydraulics.

using it to measure a velocity which has been determined by other means, it has been found that its indications are accurate to about one per cent. when the tubes are pointed as shown in Fig. 142.

130. Gauging streams — To gauge a stream is to determine the volume of water discharged by the stream per second. This determination depends upon the measurement of the sectional

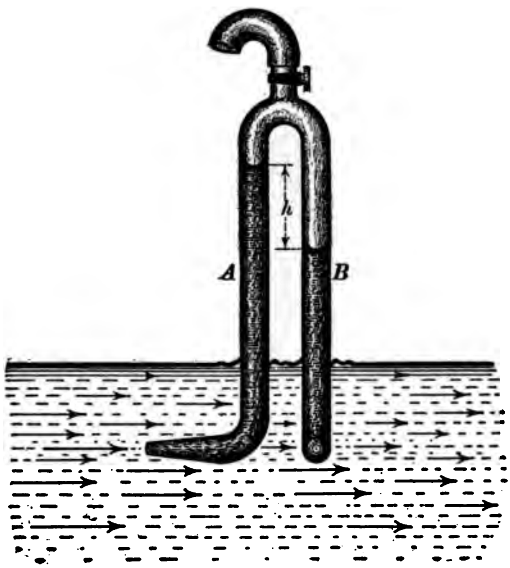


Fig. 142.

area a of the stream and of the mean velocity v of the stream, and the discharge rate of the stream is equal to av according to equation (63).

Small streams are usually gauged by means of an orifice in a temporary dam.* Let x be the distance of the center of the orifice beneath the surface of the water in the dam, then the velocity of efflux would be equal to $\sqrt{2gx}$ if the water were frictionless, and the product of this velocity and the area of the ori-

* The arrangement called a *weir* is a notch in the top of a temporary dam, and the formulas for calculating the discharge rate over a weir may be found in any treatise on Hydraulics.

fice a would be the discharge rate if the flow in the orifice were lamellar. Experiments show that the mean velocity of a water jet flowing from a sharp edged orifice like that shown in Fig. 132 is about 0.98 of the value, $\sqrt{2gx}$, corresponding to ideal frictionless flow; and experiment shows that the cross-sectional area of jet at a short distance from the orifice (where the flow becomes approximately lamellar) is about 0.62 of the area of the orifice, provided the orifice has sharp edges and is in the middle of a flat wall. Therefore the rate of discharge from an orifice like that shown in Fig. 132 is approximately equal to $0.98 \times 0.62 \times a\sqrt{2gx}$.

A large river is gauged by determining the cross-sectional area of the river and measuring the velocity of the water at a large number of points in the section so as to determine the average velocity. The velocity of the current is sometimes measured by means of floats, sometimes by means of Pitot tubes, and sometimes by means of a so-called current meter which consists of a rotating wheel like a screw propeller which drives a speed counting device. The current meter has to be calibrated by observing its speeds when it is dragged through still water at various known velocities.

131. Fluid Friction. — The dragging forces which oppose the motion of a body through the air or water, and the dragging forces which oppose the flow of fluids through pipes and channels are due to a type of friction which is called *fluid friction.*

Friction of fluids in pipes and channels. — There are two fairly distinct actions which are involved in the friction of fluids in pipes and channels, and, although these two actions always exist together, it will be instructive to consider two extreme cases in which the two actions are approximately separated.

Viscous Friction. — When a fluid flows through a very small, smooth-bore pipe, the loss of pressure is proportional to the rate of discharge, or to the mean velocity of flow of the fluid in the pipe. This fact was first established by Poiseuille (1843). In fact, for this case, we have

$$p = \frac{8\eta l Q}{\pi R^4} \tag{70}$$

in which l is the length of the tube in feet, R is the radius of its bore in fractions of a foot, Q is the volume of liquid in fractions of a cubic foot discharged per second, and η is a constant called the *coefficient of viscosity* of the liquid. It is evident from this equation that the loss of pressure due to viscous friction is very small indeed when the radius R of the tube is moderately large. In fact, viscous friction is nearly always negligible under practical conditions. A full discussion of equation (70) and a definition of the coefficient of viscosity are given in Arts. 132 and 133.

*Eddy Friction.** — Consider a series of chambers, $ABCD$, Fig. 143, communicating with each other through narrow orifices, and

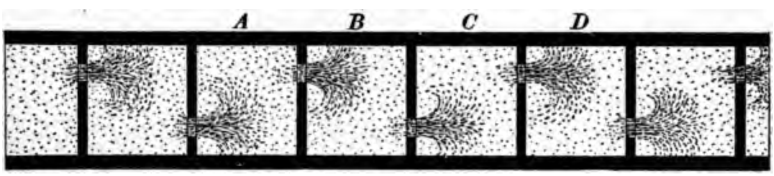


Fig. 143.

let us suppose water to flow through this series of chambers. As the water enters an orifice it gains a certain amount of velocity v, and decreases in pressure by the amount $1/2g \times dv^2$, according to Art. 127. The velocity so gained is lost by eddy action in the next chamber, and when the water flows through the next orifice it must gain velocity anew and suffer a corresponding drop in pressure, as before. It is therefore evident that *the drop of pressure through the series of chambers, $ABCD$, is proportional to the square of the rate of discharge.* This law of eddy friction is verified by experiment for a series of chambers as shown in Fig. 143, where the eddies are definitely *localized.* In an ordinary pipe, however, there is a tendency for the eddy movements to become finer grained, as it were, with increasing velocity; that is, with increased velocity a given particle of fluid acquires velocity and loses it again an increased number of times in traveling a

* A fluid entirely devoid of viscosity would not form eddies, so that all fluid friction is due to viscosity directly or indirectly.

given distance. The consequence of this fact is that the loss of pressure due to eddy friction increases more rapidly than in proportion to the square of the rate of discharge.

In all ordinary cases of the flow of fluids through pipes and channels, eddy friction is very much larger than viscous friction, and the practical formula for calculating the loss of pressure due to the flow of a fluid through a given length of pipe of a given size is based upon the assumption that the loss of pressure is proportional to the density of the fluid and to the square of the rate of discharge, or indeed, to the square of the velocity of the fluid if the pipe is of uniform size.

The meaning * of the practical formula may be made clear by the following argument: The flow of a fluid over a surface such as the interior walls of a pipe is retarded by a force which is approximately proportional to the area of the surface, to the density of the fluid and to the square of the velocity at which the fluid is flowing. Therefore we may write

$$F = kadv^2$$

in which a is the area of the surface in square feet, d is the density of the fluid in pounds per cubic foot, v is the velocity of flow in feet per second, and F is the retarding force in pounds-weight. The quantity k is called the *coefficient of friction* of the moving fluid against the walls of the pipe. It depends greatly upon the degree of roughness of the walls; and, for a given degree of roughness, it is not strictly constant, that is to say, the friction is not exactly proportional to the square of the velocity.

Consider a pipe of which the length is l feet, and the inside diameter D feet. The total area of interior walls of this pipe is πDl square feet, so that, using πDl for a in the above equation, we have $F = k\pi Dldv^2$ for the total retarding force acting on a fluid of density d flowing through the pipe at velocity v. This retarding force is equal to the difference of pressure at the two

* The formula is not rational, it is empirical, and the only thing to be done in connection with it is to exhibit its meaning clearly.

ends of the pipe multiplied by the sectional area of the bore of the pipe. Therefore, using p for the loss of pressure due to friction in the pipe, we have

$$p \cdot \frac{\pi D^2}{4} = k\pi D l d v^2$$

whence

$$p = \frac{4kldv^2}{D}. \tag{71}$$

The presence of elbows and valves causes excessive eddies and therefore an excessive loss of pressure by friction. Methods for estimating the effects of elbows and valves are given in standard works on hydraulics.

Example 1. An iron pipe one foot in diameter and 10,000 feet long discharges 4.25 cubic feet per second of water when the pressure at one end is 6,000 pounds per square foot greater than the pressure at the other end. A discharge of 4.25 cubic feet per second corresponds to a velocity of 5.41 feet per second in the pipe ($= v$). The density of the water is $62\frac{1}{2}$ pounds per cubic foot ($= d$). Substituting these values in equation (71) and we find for the coefficient k the value 0.000082.

Example 2. Compressed air at a mean pressure of 5.42 atmospheres (density 0.406 pounds per cubic foot) is forced through 15,000 feet of pipe 8 inches inside diameter at a velocity of 19.32 feet per second with a difference in pressure of 5.29 pounds per square inch between the two ends of the pipe. Reducing these data to the units employed in this chapter and substituting in equation (71), we have for the coefficient k the value 0.0000557.

These two examples indicate the method of determining the approximate value of the coefficient k under given conditions. When the value k has been so determined, equation (71) may be used for determining the amount of pressure required to force a given fluid through a given pipe at given velocity, or it may be used to determine the velocity at which a given pressure will force a given fluid through a given pipe. The value of the coefficient k is always open to question, inasmuch as it depends

greatly upon the degree of roughness of the walls of the pipe, and, in fact, it depends upon the size of the pipe and upon the velocity of the fluid; it varies from about 0.00016 for new iron pipes 0.1 foot inside diameter, to about 0.00005 for new iron pipes six feet in diameter, for velocities of three or four feet per second.

Resistance of ships. — The forces which oppose the motion of a ship depend upon two distinct actions which are called *wave friction* and *eddy friction* respectively; viscous friction, in the sense in which the term is defined above, is entirely negligible.

Wave friction. — The motion of a boat causes a heaping up of the water at the bow and the production of a hollow at the stern. The former exerts a backward pressure on the boat, and the latter causes the forward pressure on the stern to be less than it would be if the water closed in at the stern without changing its level. These changes of level at the bow and stern give rise to waves, and the work that is done in driving the boat forward in opposition to the forces due to changes of level at the bow and stern is carried away or "radiated" by these waves. Wave friction frequently amounts to more than half of the total friction of a boat at high speeds.

Eddy friction. — Anyone who has watched the water over the side of a rapidly moving steamer will have noticed next to the hull a very turbulent layer of water several inches in thickness. This turbulence involves, of course, a loss of energy and therefore a frictional drag upon the boat. This frictional drag is generally called *skin friction.* There is also, near the bow and stern, regions of turbulence which extend to some distance from the boat, especially if the boat is faulty in shape, and the frictional drag associated with this extended turbulence is usually called *eddy friction* by marine engineers.

The friction which opposes the motion of a boat is always expressed in terms of the actual force with which the water drags backward on the boat, whereas the frictional opposition to the flow of a fluid through a pipe is usually specified in terms of the difference of pressure at the two ends of the pipe due to the friction.

132. Definition of the coefficient of viscosity of a fluid. — Consider a thin layer of fluid of thickness x between two flat plates AA and BB as shown in Fig. 144, and suppose that the plate AB is moving at velocity v as indicated by the arrows. If the fluid between the plates AB were a viscous liquid like syrup, it is

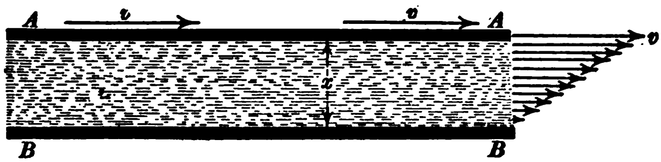


Fig. 144.

evident that a very considerable force would have to be exerted upon the plate AA to keep it in motion; in fact any fluid whatever, whether liquid or gas, is more or less like syrup in this respect, and the force F with which the motion of the plate is opposed by the fluid is proportional to its area a, to its velocity v and inversely proportional to the distance x between the plates. That is

$$F = \eta \cdot \frac{av}{x} \tag{72}$$

in which the proportionality factor η is called the coefficient of viscosity of the fluid.

Examples. — The coefficient of viscosity of water at ordinary room temperature is 0.0000215 and the coefficient of viscosity of good machine oil is about 0.00085, F, a, v and x being expressed in terms of the units specified at the beginning of this chapter. It may seem, therefore, that water would be a better lubricant than the oil, but a layer of water would quickly flow out from between a shaft and a bearing surface, whereas a rotating shaft continually carries a fresh supply of a viscous liquid like oil into the space between the shaft and the bearing surface.

133. Flow of a viscous liquid through a small smooth-bore tube. — Let R be the radius of the bore of the tube, l the length of the tube, p the difference of pressure of the liquid at the ends of the tube, and v the velocity of the liquid at a point distant r from the axis of the tube. Consider a cylindrical portion of the moving liquid of radius r and coaxial with the tube. The surface of this cylindrical portion of liquid moves as a *solid rod* through the tube at velocity v. Similarly, the cylindrical surface of radius $r + \Delta r$ moves through the tube as a *hollow shell* at velocity $v - \Delta v$. The layer of liquid between this *rod* and *shell* is under the same conditions of motion as the layer of liquid between the plates AA and BB in Fig. 144. Therefore, writing Δv for v in equation (72), writing Δr for x, and writing $2\pi rl$ for a we have

$$F = \eta \cdot \frac{2\pi rl \cdot \Delta v}{\Delta r}$$

where F is the force which must act on the end of the *rod* to overcome the viscous drag; but this force is equal to the area of the end of the rod multiplied by p, so that

$$\pi r^2 p = \eta \cdot \frac{2\pi r l \cdot \Delta v}{\Delta r}$$

or

$$\frac{dv}{dr} = \frac{p}{2\eta l} \cdot r$$

whence

$$v = \frac{pr^2}{4\eta l} + \text{a constant}$$

but when $r = R$, $v = 0$, so that the constant of integration is equal to $-pR^2/4\eta l$ and therefore

$$v = \frac{pr^2}{4\eta l} - \frac{pR^2}{4\eta l} \qquad \text{(i)}$$

The velocity at each part of the tube is thus determined. To find the volume V of fluid discharged in time τ, consider a section of the tube, Fig. 145. The velocity over all the area, $2\pi r\Delta r$, of the dotted annulus, is v, so that the volume ΔV, flowing across this annulus in time τ, is $\Delta V = 2\pi r \Delta r \cdot v \cdot \tau$. Substituting v from (i), we have

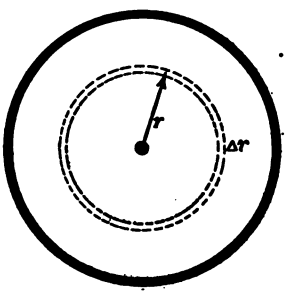


Fig. 145.

$$dV = \frac{\pi p \tau}{2l\eta} r^3 dr - \frac{\pi p R^2 \tau}{2l\eta} r dr$$

or

$$V = \frac{\pi p \tau}{2l\eta} \int_0^R r^3 dr - \frac{\pi p R^2 \tau}{2l\eta} \int_0^R r dr \qquad \text{(ii)}$$

Therefore

$$V = \frac{\pi p R^4 \tau}{8l\eta}$$

PROBLEMS.

156. Find the mean velocity at which water must flow in a canal 20 feet wide and 6 feet deep, in order that the rate of discharge may be 500 cubic feet per second.

How many acres of storage basin would be required to store an amount of water sufficient to maintain this flow of water for 24 hours, the average depth of the water in the storage basin to be 10 feet?

157. The density of gas in a steady stream is 0.20 pound per cubic foot at one point and 0.35 pound per cubic foot at another point. The section of the stream is 0.25 square inch at the first point and 0.56 square inch at the other point. Find the ratio of the velocities of the stream at the two points.

158. How much work in foot-pounds is required to pump 10,000 cubic inches of water into a reservoir in which the pressure stands at the constant value of 150 pounds per square inch above atmospheric pressure?

159. The velocity of a water jet is 200 feet per second, what is the kinetic energy of the water in foot-pounds per cubic inch? One cubic inch of water weighs 0.0376 pound.

160. Calculate the velocity of efflux of kerosene from a vessel in which the pressure is 52 pounds per square inch above atmosphere pressure. The density of kerosene is 0.03 pound per cubic inch.

161. Water flows in a 12-inch main at a velocity of 4 feet per second and encounters a partly closed valve through which the section of the stream is reduced to 0.36 square foot. Calculate the loss of pressure at the valve due to friction.

Note. — As the water enters the narrow passageway in the valve, its velocity increases by a definite amount, and its pressure falls off accordingly, as explained in Arts. 127 and 128. As the water issues from the narrow passageway, it retains its velocity as a jet flowing through the surrounding water, so that its pressure does not rise again, and the excess of velocity is then destroyed by eddy action. Therefore the loss of pressure through the valve is approximately equal to the drop of pressure due to the increased velocity of the water as it enters the narrow passage.

When a portion of a moving fluid meets with another portion which is moving at a less velocity, the excess of velocity is lost in eddy motion and we have what is frequently called a "shock."

162. A street water-main 7 inches inside diameter has a throat 3 inches in diameter inserted in it. The flow of water through the pipe is 1½ cubic feet per second and the pressure in the 7-inch pipe is 90 pounds per square inch. What is the pressure in the throat in pounds per square inch, ignoring friction?

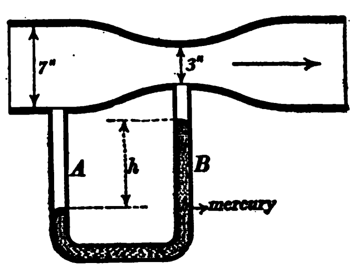


Fig. 163*p*.

163. The difference in level, *h*, Fig. 163*p*, is observed to be 6 inches. Calculate the rate of discharge of water through the pipe in cubic feet per second.

Note. — The specific gravity of mercury is 13.6, and the tube AB is entirely filled with water above the surface of the mercury.

164. A pair of Pitot tubes is placed in the stream of air issuing from a fan blower, as shown in Fig. 164*p*, and the difference in level, h, is observed to be $2\frac{1}{2}$ inches, the tubes being filled with water. Calculate the velocity of the air stream.

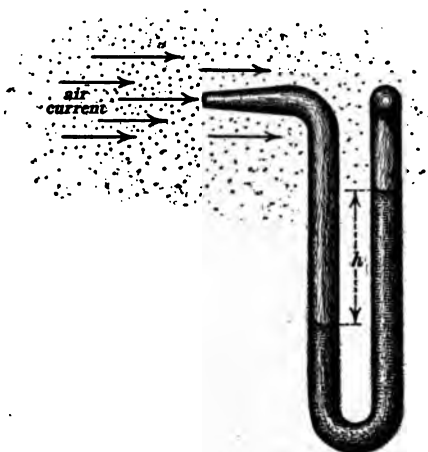


Fig. 164*p*.

Note. — In this problem ignore the compressibility of the air, and assume its specific gravity to be 0.08 pound per cubic foot.

165. A temporary dam made of thin boards has a circular hole 1 foot in diameter cut through it, and the water rises to a height of $1\frac{1}{2}$ feet above the center of the hole. Calculate the discharge rate of the stream in cubic feet per second.

166. A supply of 10 cubic feet per second of water is to be brought from a reservoir to a center of distribution in a small city. The height of the surface of the reservoir above the point of distribution in the city is 350 feet and it is desired to have an available head of 150 feet at the center of distribution. Required the size of pipe that is necessary to deliver the water, the length of pipe being 16,000 feet.

167. A water tank is installed for the protection of a factory against fire. The water level in the tank is 75 feet above a certain fire hydrant in the building. The pipe leading from the tank to the fire hydrant consists of 150 feet of 4-inch pipe in which there is one Pratt and Cady check valve, and four short-turn ells; and 200 feet of 3-inch pipe in which there are three short-turn ells. Find the number of gallons of water per second that

can be delivered at the hydrant allowing a loss of head of 25 feet in the pipe.

Note. — A four-inch Pratt and Cady check valve has a resistance equivalent to 25 feet of four-inch pipe, and short-turn ells each have a resistance equivalent to 4 feet of pipe of the same size.

CHAPTER XI.

WAVE MOTION AND OSCILLATORY MOTION.

134. Wave motion as a basis of certain branches of physical theory. Scope of this chapter. — Many of the fundamental principles of mechanics, such as Newton's laws of motion and the principle of the conservation of energy, are used throughout the whole range of the physical sciences. Indeed, the principle of the conservation of energy is so widely used in the study of physics and chemistry that its purely mechanical origin * is generally lost sight of. There is, however, another aspect in which mechanics is utilized in connection with general physics, inasmuch as many physical and chemical theories are essentially mechanical. Thus, the molecular theory in chemistry is as truly a mechanism as a locomotive, except that a locomotive must be actually constructed to be used, whereas the molecular theory need only be imagined.†

The wave theory. — That group of ideas which relates to the kind of mechanical action called wave motion, is perhaps more useful in general physics than any other group of mechanical ideas, not excepting even the group of mechanical ideas which is called the molecular theory. Nearly every phenomenon of sound and light, and nearly all of the phenomena of oscillatory motion become intelligible in terms of the ideas of wave motion.

Complexity of water waves. — In undertaking to establish the more important ideas of wave motion we are confronted with a serious difficulty, namely, that water waves, the only kind of

* See Art. 59.

† The student should guard against the idea that it is easier to imagine the molecular theory than it would be to build a locomotive, for this is by no means the case. A million men, perhaps, can be found capable of building a locomotive that will work, where one can be found who can imagine the molecular theory so that it will work; the final test is the same in both cases: will the thing work?

waves with which every one is familiar, are excessively complicated; invisible sound waves in the air and the even more intangible light waves in the ether, in their more important aspects, at least, are extremely simple in comparison. The wave theory, however, originated in the application, to sound and light, of the ideas which grew out of a familiarity with the behavior of water waves, and, in attempting to establish the wave theory, one is obliged to base it upon the familiar phenomena of wave motion as exemplified by water waves.

Wave media. — The material or substance through which a wave passes is called a medium. Thus, the air is the medium which transmits sound waves, and the ether is the medium which transmits light waves. A wire (or a rod) may be spoken of as a medium when one is concerned with the transmission of waves along the wire (or the rod). During the passage of the wave the medium always moves to some extent, but the velocity with which the medium actually moves is generally very much less than the velocity of progression of the wave.* Thus, when the end of a long rope is moved rapidly to and fro sidewise, waves travel along the rope and each point of the rope oscillates as the waves pass by. In some cases the medium is left permanently displaced after the passage of the wave; in other cases, the medium returns to its initial position after the passage of the wave.

Wave pulses and wave trains. — When a stone is pitched into a pond a wave emanates from the place where the stone strikes. When a long stretched wire is struck sharply, with a hammer, a single wave (a bend in the wire) travels along the wire in both directions from the point where the wire is struck. When a long steel rod is struck on the end with a hammer, a single wave (an endwise compression of the rod) travels along the rod. When an explosion takes place in the air, the firing of a gun, for example, a single wave (a compression of the air) travels outwards from the

* The mathematical theory of wave motion is developed on the assumption that the actual velocity v of the medium is very small in comparison with the velocity of wave progression V.

explosion. Such isolated waves are called *wave pulses*. When a disturbance at a point in a medium is repeated in equal intervals of time, the disturbance is said to be *periodic*. Such a disturbance sends out a succession of similar waves constituting what is called a *wave train*.

The wave pulse involves all * of the important mechanical actions of wave motion, and the mechanics of wave motion can be developed in the simplest possible manner by considering the behavior of wave pulses.

The applications of the wave theory to sound and light depend very largely upon a consideration of wave trains, but the theory of wave trains does not have any important applications in mechanics. This chapter is limited therefore to the study of wave pulses.

Wave shape. — A term which is frequently used in the discussion of wave motion is the term wave shape, and the meaning of this term may be best explained by considering wave motion along a stretched wire. Let us consider first an entirely general case as follows: Let AB, Fig. 146, be a wire under a tension of T

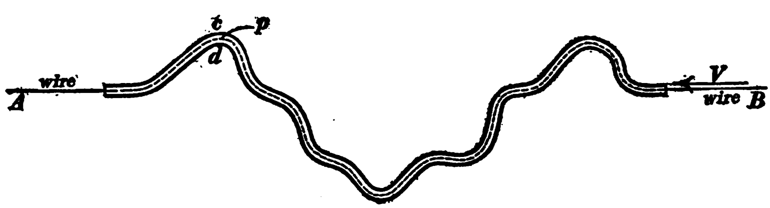


Fig. 146.

pounds-weight, and suppose that each foot of the wire weighs m pounds. Imagine the wire to be drawn at a velocity of V feet per second through a crooked tube, WW, and let it be assumed that the wire slides through the tube without friction. Then the

* The mechanical action which serves as a basis for the theory of the dispersion of light depends upon the effect a wave train has upon the particles of the medium where these particles have a tendency to oscillate at a definite frequency, but a discussion of this effect is beyond the scope of an elementary text.

wire will not exert any force against the sides of the tube, if the velocity V satisfies the equation

$$V = \sqrt{\frac{gT}{m}} \tag{73}$$

and at this velocity the moving wire would therefore retain its crooked (stationary) shape if the tube could be removed. This tendency of a bend, once established in a rapidly moving flexible wire or chain, to persist is strikingly illustrated by the behavior of the loose chain of a differential pulley * when the pulley is rapidly lowered and the loose chain set into rapid motion, and a series of stationary bends is often seen in a rapidly moving belt.

The absence of force action between tube and wire in Fig. 146 may be shown as follows: Consider any point, p, of the tube. The portion of the tube in the immediate neighborhood of this point is necessarily a portion of a circle of a certain radius, r. Therefore, if the wire were stationary, its tension would produce a force against the side, d, of the tube, and this force would be equal to T/r pounds per foot of length of wire, according to Art. 40. But to constrain the particles of the wire to the small circular arc at p, an unbalanced force equal to $mV^2/(rg)$ pounds-weight per foot must act on the wire, pulling it towards the side, d, of the tube, according to Art. 38. Therefore, when $T/r = mV^2/(rg)$ or when $V = \sqrt{gT/m}$, the side force due to the tension of the wire is just sufficient to constrain the particles of the wire to the curved path at p, whatever the curvature at that point may be, and no force need act upon either side of the tube.

We may imagine everything in Fig. 146 (moving wire and tube) to be set moving to the right at a velocity equal and opposite to the velocity at which the wire is moving to the left. The result would be that the wire would be stationary and the tube would move to the left at velocity V, and, of course, the wire would pass through the tube without exerting any force upon the

* See figure illustrating problem 84 on p. 126.

sides of the tube, as before, so that the bend, WW, would continue to move along the wire without changing its shape, even if the tube were non-existent. Such a moving bend constitutes a wave, and the only motion of a given point of the wire during the passage of the wave would be its sidewise motion. *The term wave shape refers to the distribution of the velocity of the medium (sidewise velocity of the wire in Fig. 146) in a wave.* This matter may be made clear by the following example. Imagine the straight tube, WW, Fig. 147, to slide along the wire, AB, at velocity V, thus producing a wave. The wire at each point in

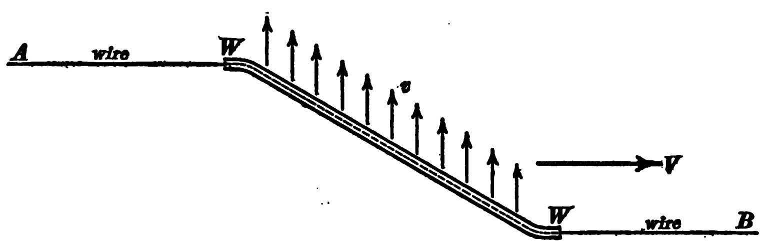


Fig. 147.

the tube is moving sidewise at a constant velocity as indicated by the small arrows, and the portion of the wire in the tube is uniformly stretched. The uniform stretch of the wire in the tube in Fig. 147 is evident when we consider first that the horizontal component of the tension of the wire in the tube is equal to the tension, T, of the portions of the wire beyond the tube, so that the tension of the wire in the tube is greater than T, and second that the tube is straight so that the tension of the wire in the tube is uniform.

In discussing waves it is convenient to draw the line, AB, Fig. 148, in the direction of progression of the wave and to represent the actual velocity of the medium at each point in the wave by an ordinate, y, as shown; or to represent the actual stretch (or compression) of the medium at each point by the ordinate, y. Thus, the wave shown in Fig. 147 would be represented by the rectangle in Fig. 148, inasmuch as the side

velocity of the wire and the stretch of the wire are everywhere constant in value between WW. The wave represented in Fig. 147 is therefore called a rectangular wave, or a rectangular wave pulse. *The entire discussion of wave pulses in Arts. 135 to 138 refers to rectangular wave pulses,* because rectangular wave pulses are the simplest to describe.

*Pure and impure waves.** — In certain cases a wave pulse retains its shape as it travels through a medium. Thus, for example, a bend travels along a stretched flexible string or wire with unchanging shape. Such a wave is called a *pure wave.* In other cases, a wave spreads out more and more as it travels

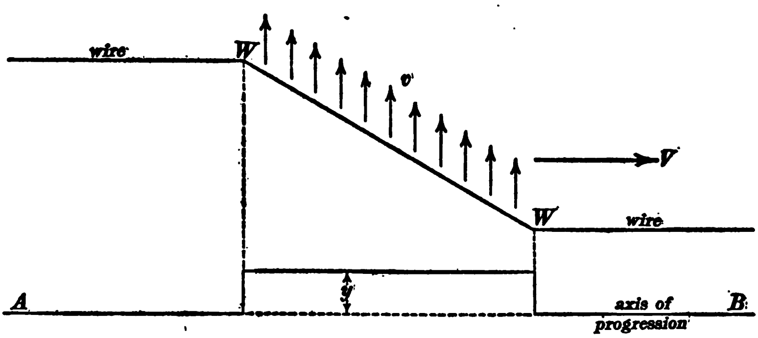


Fig. 148.

through a medium. Thus, for example, a sharply defined water wave in a canal spreads out over a greater and greater length of the canal and becomes less sharply defined as it moves along. Such a wave is called an *impure wave.* Waves are generally rendered impure by friction or by the imperfect elasticity of a medium. This matter is discussed very briefly in Art. 135.

The distinction between pure and impure waves is very important in telephone engineering. The transmission of articulate

* The best discussion of the difference between pure and impure waves is that which is given by Heaviside in his Electro-Magnetic Theory, Vol. I, pages 307 to 466. As an introduction to this discussion by Heaviside, the student should read an article on Electric Waves and the Behavior of Long Distance Telephone Lines, by W. S. Franklin, Journal of the Franklin Institute, July, 1905.

speech by the telephone depends upon the passage of sharply defined electric waves along the telephone wires. If these waves remain pure, every detail of shape is retained as the wave moves along, whereas, if the waves become impure, each part of every wave begins to spread over the adjoining parts and the result is that the fine details of shape become obliterated. An ordinary telephone line converts the pure electric waves which start out from the sending station into impure waves, thus tending to destroy the sharpness of detail and making it impossible to transmit articulate speech if the line is long; whereas a telephone line that is "loaded" * tends to keep the waves pure so that details of shape are retained as the wave travels over the line, and clear articulate speech can be reproduced at the distant end of the line.

135. Wave pulses in a canal. — A consideration of the simplest

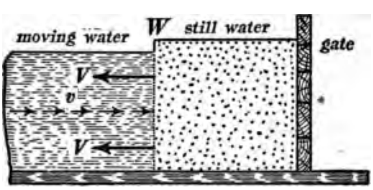


Fig. 149.

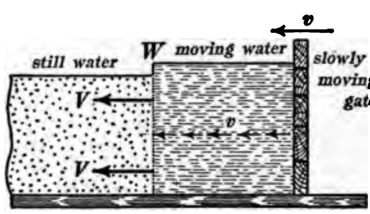


Fig. 150.

kind of wave motion in a canal, namely, the kind in which the only perceptible motion of the water in the wave is a uniform horizontal flow, will serve better than anything else as an introduction to a discussion of wave motion along rods, air columns, and stretched strings.

The simplest basis for the discussion of the propagation of a wave along a canal is as follows. Water flows along a canal of rectangular section at a depth of x feet and at a uniform velocity (small) of v feet per second. A gate is suddenly closed; the moving water, in being brought to rest against the gate, heaps

* A "loaded" telephone line is a line in which coils of wire wound on iron cores are inserted at intervals. This arrangement is due to Heaviside and to Pupin.

up to a definite depth $x + h$, as shown in Fig. 149; and a *wave of arrest*,* W, moves along the canal at a definite velocity V. The water at a given point in the canal continues to move with unchanged velocity until the *wave of arrest* reaches that point, when the water suddenly comes to rest and heaps up to the height $x + h$. The velocity of propagation of the *wave of arrest* W is

$$V = \sqrt{gx} \tag{74}$$

where V is expressed in feet per second, g is the acceleration of gravity, and x is the depth of the water in the canal in feet. It is interesting to note that the velocity V of the wave is the velocity that would be gained by a body falling freely through the distance $x/2$.

Proof of equation (74).—Let b be the breadth of the canal. Consider a transverse slice of water one foot thick. The volume of this slice is bx cubic feet, and its mass is dbx pounds, where d is the density of the water in pounds per cubic foot. Therefore the kinetic energy of this slice of water when it is moving at a velocity of v feet per second is $1/2g \times dbxv^2$, according to equation (27). When the *wave of arrest*, W, Fig. 149, reaches the slice of water under consideration, the slice, as it comes to rest, is squeezed together and increased in depth to $x + h$. The slice is decreased in thickness in proportion to its increase in depth, so that its thickness is reduced to $x/(x + h)$, or $1 - h/x$, of a foot, since h is very small. Therefore the decrease of thickness is h/x of a foot. The force acting to reduce the thickness of the slice is to be considered as that force which is due to the *increase* of pressure in the water produced by the *increasing* depth, h. This increase of pressure is equal to hd when the slice has reached its greatest height, so that the average increase of pressure due to increasing depth is $\frac{1}{2}hd$, which produces over the face of the slice a force equal to $\frac{1}{2}hd \times bx$, and the product of this force and the decrease of thickness of the slice gives the work done in decreasing its thickness. Therefore, since this work is equal to the original kinetic energy of the slice, we have:

* The idea of the *wave of arrest* and of the *wave of starting* is so important in the discussion of wave pulses that it is worth while to illustrate it as follows. Imagine a troop of soldiers to be marching along in single file at a distance of three feet apart, and imagine every soldier to continue to march as long as there is space in front of him. If the front man in the troop is suddenly stopped, the other men are stopped in succession as they come against each other, and the stopping occurs at a point which travels uniformly from the front to the rear of the colum, a *wave of arrest*, as it were. If the front man then starts, the other men in the column start in succession, and the starting occurs at a point which travels uniformly from the front to the rear of the column, a *wave of starting*, as it were.

$$\frac{1}{2g} \cdot dbxv^2 = \tfrac{1}{2}dbh^2$$

or

$$v^2 = \frac{gh^2}{x} \qquad \text{(i)}$$

Consider the instant t seconds after the closing of the gate in Fig. 149. The *wave of arrest W* has reached a distance Vt from the gate, and the excess of water that is represented by the raising of the water level ($= Vt \times h \times b$ cubic feet) is the amount of water supplied by the flow of the canal in t seconds ($= bxvt$ cubic feet). Therefore

$$Vthb = bxvt$$

or

$$V = \frac{xv}{h} \qquad \text{(ii)}$$

whence, substituting the value of v from equation (i) in equation (ii), we have equation (74).

Precisely the same action that is represented in Fig. 149 may be produced in a still-water canal by moving the gate along the canal like a piston at a low velocity, v, as indicated in Fig. 150.

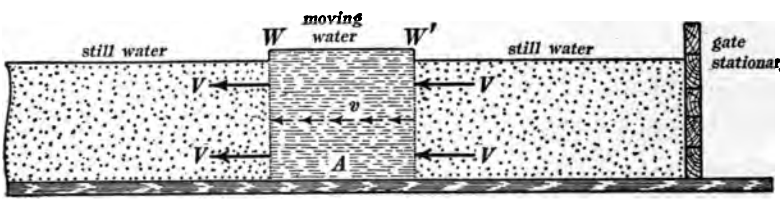


Fig. 151.

The water, in being set in motion, heaps up to a definite depth $(x + h)$, and a *wave of starting*, W, moves along the canal at a definite velocity, V. If the gate is suddenly stopped, the *wave of starting*, W, continues to move along as before, the water next to the gate, in being stopped, drops to its normal depth, x, and a *wave of arrest*, W', moves along the canal as shown in Fig. 151.

The uniformly moving and uniformly elevated body of water, A, constitutes what is called a *complete wave* or simply a *wave*, the water in front of the wave is continually set in motion at velocity v and raised to the depth $x + h$, the water in the back part of the wave is continually brought to rest and lowered to the

normal depth, x, and thus the state of motion which constitutes the wave A travels along the canal without changing its character, friction being neglected.

An essential feature of a wave which moves along a canal without changing its shape, is that *the kinetic energy due to the uniform velocity v is equal to the potential energy due to the elevation h.* When this relation obtains the wave is called *a pure wave*, and when this relation does not obtain the wave is called an *impure wave*.

The behavior of an impure wave pulse in a canal may be understood by considering an extreme example of an impure wave as follows: Consider an elevated portion of still water in a canal, as shown in Fig. 152. This body of elevated water is an

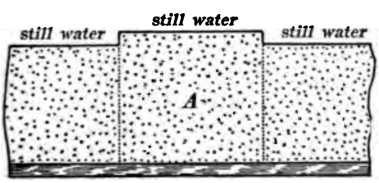


Fig. 152.

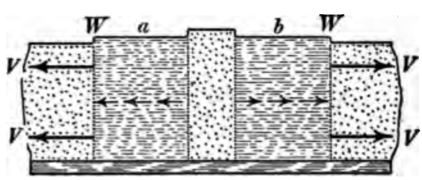


Fig. 153.

impure wave inasmuch as its velocity of flow, v, is zero, and therefore the potential energy of elevation is of course not equal to the kinetic energy of flow. Such an elevated portion of still water breaks up into two oppositely moving pure waves, and the initial stage of this process of breaking up is indicated in Fig. 153.

When a wave like A, Fig. 151, travels along a canal, the velocity of flow, v, is decreased by friction, whereas there is no action tending to reduce the elevation h. The result is that the portion of the elevation which is in excess of what is required to constitute a pure wave with what remains of the velocity of flow, behaves exactly like the elevation A in Fig. 152, that is, this excess of elevation breaks up into two pure waves a and b, Fig. 153, the portion a merges with the original wave A, and the portion b shoots backwards.

Figure 154 represents on an exaggerated scale, a pure wave, *A*, started at a given point on a canal. The velocity of flow, v, in this wave is continually reduced by friction as the wave travels along, and the excess of elevation which is being thus continually left in the wave causes a long-drawn-out wave to shoot backwards,

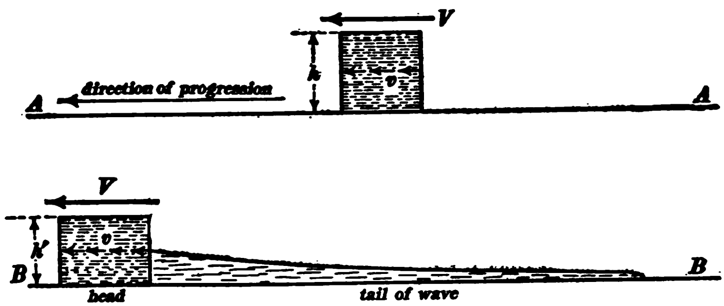


Fig. 154.

and after a time the wave takes on the form shown by *BB*. The tail of the wave extends far back of the original starting point of the wave.

If a canal is brimful of water so that the elevation causes an overflow or spill, the tendency is for a wave to remain pure and therefore to be propagated without change of shape, because the

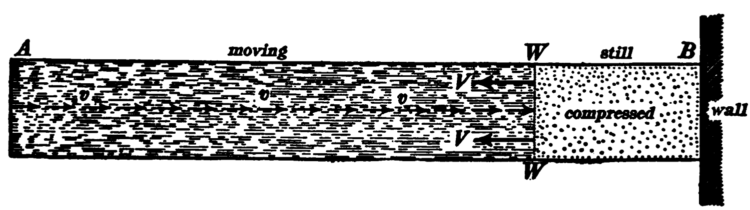


Fig. 155.

elevation is reduced by spill and the velocity of flow v is reduced by friction. This is precisely analogous to the action which takes place on a poorly insulated telephone line and causes such a telephone line to transmit speech more distinctly than the same line would if it were thoroughly insulated.

136. Wave pulses along a steel rod.—Before proceeding to discuss the production of a complete wave of compression (or stretch) along a steel rod, it is of interest to consider the behavior of a steel rod when it is moving endwise at a small velocity, *v*, and strikes a rigid wall, as shown in Fig. 155. The portion of the rod next to the wall is compressed and immediately brought to rest, and a *wave of arrest*, *W*, moves along the rod, as indicated.

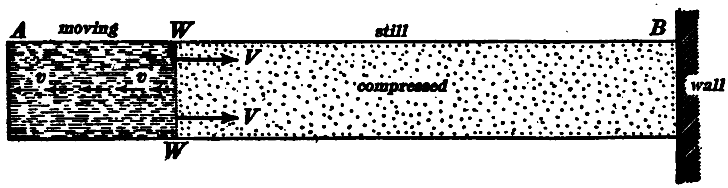


Fig. 156.

At the instant that the *wave of arrest* reaches the free end, *A*, of the rod in Fig. 155, the entire rod is stationary and uniformly compressed. Therefore balanced forces act on every portion of the rod except the layer of material at the extreme end, *A*, so that this layer immediately starts to move at reversed velocity, *v* (away from the wall), as indicated in Fig. 156, and a *wave of starting* travels back along the rod, as indicated in the figure.

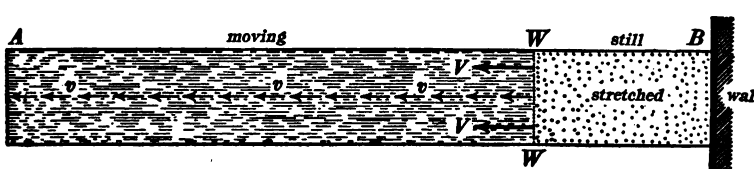


Fig. 157.

When this *wave of starting* reaches the end *B* of the rod, the entire rod is free from compression and is moving away from the wall at velocity, *v*.

If the rod be glued fast, as it were, to the wall, the end *B* of the rod is immediately brought to rest and stretched, and a *wave of arrest* travels along the rod as shown in Fig. 157. When this *wave of arrest* reaches the free end, *A*, of the rod, the entire

rod is stationary and uniformly stretched, so that the end layer at A immediately begins to move towards the wall, and a *wave of starting* travels back towards the wall as indicated in Fig. 158. When this wave of starting reaches B, the entire rod is precisely in its initial condition, namely, moving towards the wall at uniform velocity v. The action here described takes place so rapidly

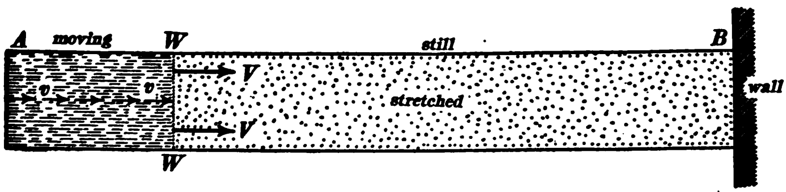


Fig. 158.

that it cannot be followed with the eye. In fact, the *wave of arrest* and the *wave of starting* in Figs. 155 to 158 travel at a velocity of 16,950 feet per second, so that the entire cycle of movements above described would take place 424 times per second in a steel rod 10 feet long, inasmuch as the *waves of arrest*

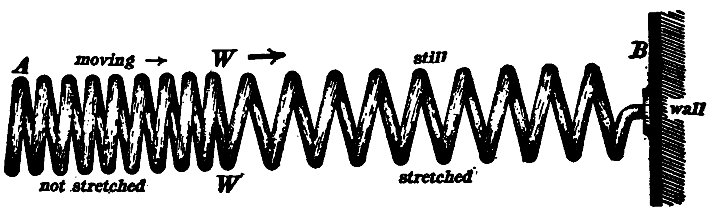


Fig. 159.

and *starting* must travel over the rod four times in one complete cycle of movements.

A helical spring may be made to perform the same series of movements as the steel rod as above described, and at a much slower rate. Thus, if one end of a helical spring be attached to a wall, as shown in Fig. 159; the spring may be uniformly stretched by pulling on the end A. If the spring is then released the free end begins to move towards the wall, and a *wave of starting* travels towards the wall exactly as described in connection

with Fig. 158; then a *wave of arrest* travels back from B to A exactly as described in connection with Fig. 155; a *wave of starting* (velocity v away from the wall) then travels from A to B; and a *wave of arrest* then travels from B to A, bringing the spring into its initial uniformly stretched condition.

This example of the traveling of waves of starting and arrest along a steel rod or a helical spring illustrates one of the most important applications of the ideas of wave motion, namely, their application to the study of oscillatory motion. In the discussion of the oscillatory motion of a weight fixed to the end of a spring (Art. 43), and in the discussion of the oscillatory motion of a torsion pendulum (Art. 67), the elastic part of the system is assumed to have negligible mass, and the massive part of the system is assumed to have negligible elasticity, or, in other words, the mass is assumed to be *concentrated* in one part of the oscillating system and the elasticity is assumed to be *concentrated* in another part of the oscillating system. The oscillatory motion of such a system is extremely simple in comparison with the infinite variety of oscillatory movements that may be performed by a given * system in which elasticity and mass are distributed throughout the system.†

The velocity of propagation of the *waves of arrest* and *starting* in a steel rod may be derived as follows: After a certain interval of time t the *wave of arrest* W, reaches the end A in Fig. 155, having traveled over the length of the rod L. During this time, t, the end B of the rod has been stationary and the end A has been moving steadily at velocity v. Therefore the rod has been shortened by the amount $l = vt$, so that the shortening per unit length $l/L[=\beta$ of equation (50)] is equal to vt/L. The kinetic energy in foot-pounds of the moving rod before it strikes the wall is $1/2g \times Lad \times v^2$, according to equation (27), where d is the density of the rod in pounds per cubic foot and a is its sectional area in square feet. At the instant that the *wave of arrest* reaches the end A, the potential energy of the compressed rod is $\frac{1}{2}E\beta^2 \times La$, according to equation (50), where E is the stretch modulus of the

* See Art. 142.

† The mode of treatment of an oscillating system with concentrated mass and concentrated elasticity is exemplified by the simple theory of alternating currents (concentrated inductance and capacity), and the mode of treatment of an oscillating system with distributed mass and distributed elasticity in terms of wave motion is exemplified by the theory of transmission lines for alternating currents (distributed inductance and distributed capacity.).

steel in pounds per square foot ($= 144 \times 30{,}000{,}000$), and La is the volume of the rod. This potential energy of compression is equal to the original kinetic energy of the rod so that

$$\frac{1}{2g} \cdot Ladv^2 = \frac{1}{2}E\left(\frac{l}{L}\right)^2 La = \frac{1}{2}E \cdot \frac{v^2t^2}{L^2} \cdot La$$

so that

$$\frac{L^2}{t^2} = V^2 = \frac{gE}{d}$$

or

$$V = \sqrt{\frac{gE}{d}} \tag{75}$$

where V is required velocity of progression of the *wave of arrest* in Fig. 155.

A completely self-contained wave of compression may be produced in a steel rod in a manner exactly analogous to the production of a complete water wave in a canal by moving a gate as indicated in Fig. 150. One end of a long steel rod AB, Fig. 160 is struck with a hammer, and let us assume, for the sake of

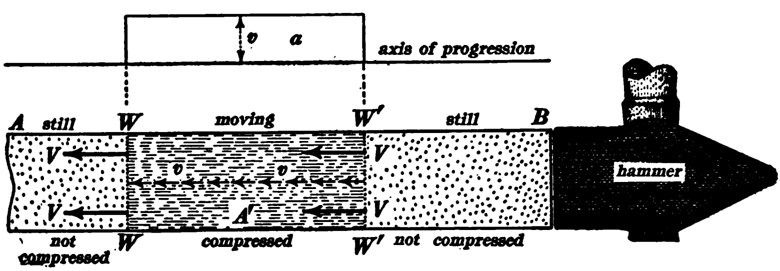


Fig. 160.

simplicity, that the hammer continues to move at a constant velocity v while it pushes on the end of the rod, and then rebounds. The result of such a hammer blow would be to set up a *wave of starting* W, the portion of the rod behind W would be moving at uniform velocity v and would be uniformly compressed. At the instant of rebounding of the hammer, however, the compression in the moving portion of the rod would cause the end layer B of the rod to stop, and a *wave of arrest* W' would follow the *wave of starting* W as indicated in the figure.

The above description applies to the production of a wave of compression in a steel rod. A sudden pull on the end B of the

rod in Fig. 160 would produce a completely self-contained wave of extension or stretch. Such a wave is shown in Fig. 161.

The complete waves $A'A'$ in Figs. 160 and 161 are represented graphically by the small rectangles, *aa*, the ordinates of which

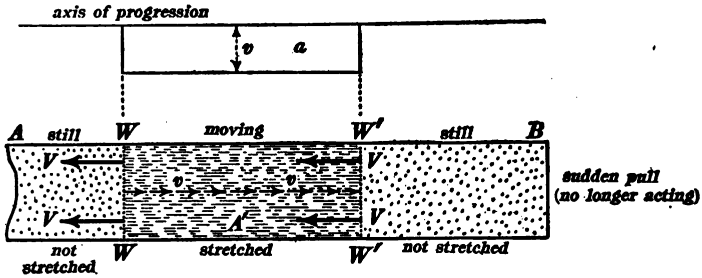


Fig. 161.

represent the uniform velocity of motion v of the portion of the rod which is in the wave.

Reflection with or without phase reversal. — An action which is very important in the theory of reflection of light and sound, is

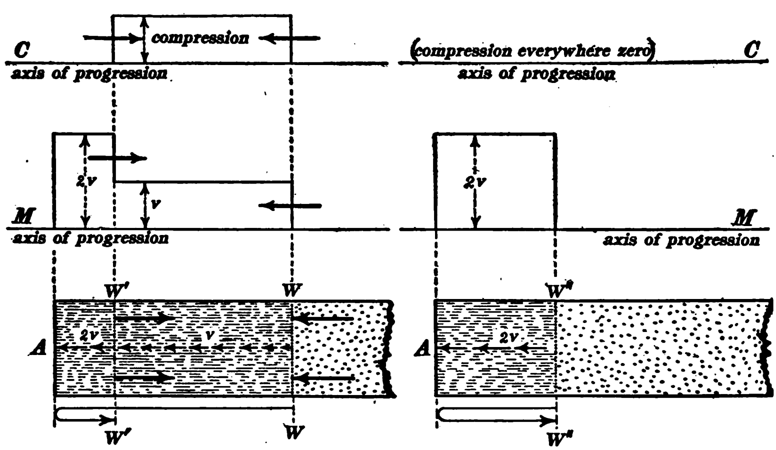


Fig. 162. Fig. 163.

the action which is called reflection with or without phase reversal, as the case may be. This action may be easily understood by considering the reflection of a complete wave like A' of Fig. 160

or Fig. 161. Let us consider first the reflection of a complete wave from the free end of a rod as shown in Figs. 162 to 165. At the moment when the front of the complete wave reaches the end of the rod, the compression, being suddenly relieved, is converted into motion of the end layer, which motion, added to the previously existing motion in the wave, causes a doubled velocity of the end layer. This doubled velocity establishes itself by a *wave of starting*, $W'W'$, as shown in Fig. 162, and the region in which this doubled velocity exists is entirely freed from compression. A moment later, when the *wave of starting* $W'W'$ and the *wave of arrest* WW in Fig. 162 become coincident,

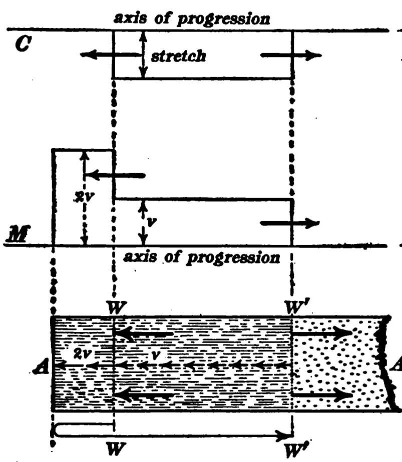


Fig. 164.

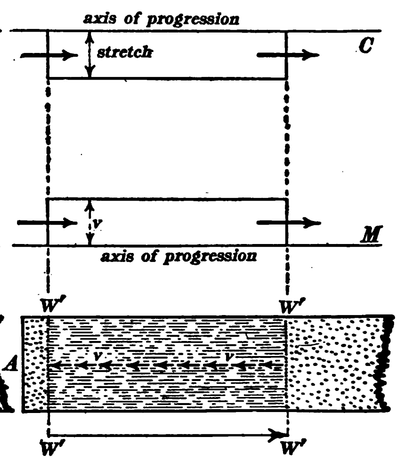


Fig. 165.

the entire energy of the original wave is represented by the doubled velocity of a portion of the rod one-half as long as the original complete wave, and the compression is everywhere zero, as shown in Fig. 163. The motion of the end of the rod as shown in Fig. 163 begins to produce a stretch in the region near $W''W''$, causing the doubled velocity on one side of $W''W''$ to be reduced to the normal value v, and setting the material to the right of $W''W''$ in motion. A wave of semi-arrest, WW, Fig. 164, travels towards the end of the rod and the *wave of*

starting, $W'W'$, travels, as indicated in Fig. 164. When the wave of semi-arrest WW, Fig. 164, reaches the end of the rod, the combined action of the uniform stretch and the uniform velocity v of the portion of the rod which constitutes the reflected (complete) wave brings the rod to rest at the extreme end and frees it from distortion as the reflected (complete) wave moves towards the right in Fig. 165. The diagrams C and M are graphical representations of the compression (or stretch) and of the motion v of the material of the rod.

The details of reflection of a complete wave of stretch or extension from the free end of a rod are very similar to the details of reflection of a complete wave of compression as described

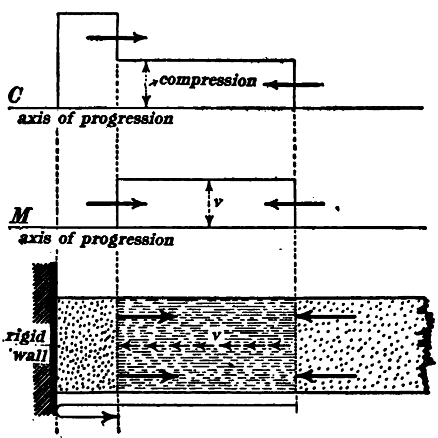


Fig. 166.

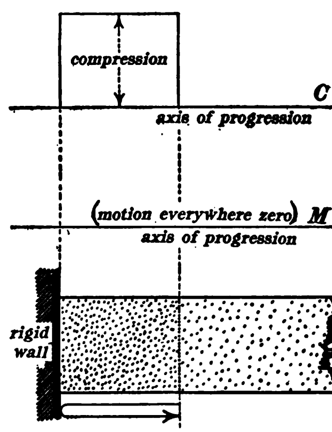


Fig. 167.

above. The reflection of either kind of a wave from the free end of a steel rod results in the conversion of compression into stretch or the conversion of stretch into compression, leaving the velocity, v, in the wave unchanged in direction. Therefore this kind of reflection is called reflection without reversal of velocity-phase (but with reversal of distortional-phase).

The details of reflection of a complete wave of compression from the end of a steel rod which rests against a rigid wall are shown in Figs. 166 to 169. When the original complete wave reaches

the end of the rod the motion in the wave compresses the end layer of the rod, which compression is added to the previously existing compression in the wave, giving a double compression in the end layer of the rod, the velocity in this layer being reduced to zero. The details of the entire action may be seen from the

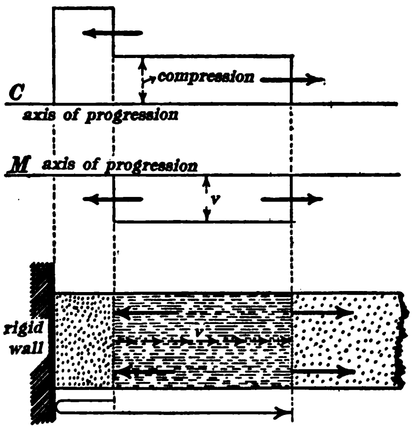


Fig. 168.

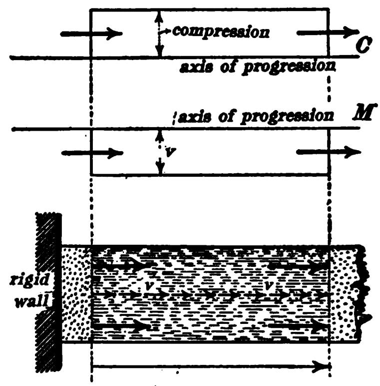


Fig. 169.

figures. The details of reflection of a complete wave of stretch or extension from the rigid end of a rod are not very different from the details of reflection of a complete wave of compression. The reflection of either kind of a wave from the rigid end of a rod results in the reversal of the velocity v, without converting com-

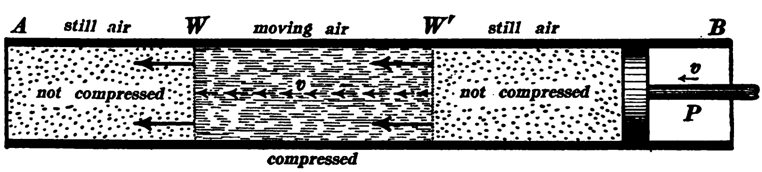


Fig. 170.

pression into stretch or stretch into compression. Therefore the reflection from the rigid end of a rod is called reflection with reversal of velocity-phase (but without reversal of distortional-phase).

137. Wave pulses in air and water pipes. Figure 170 represents a long pipe filled with air. If the piston is moved at a small

velocity v for a short time and then brought to rest, a complete wave of compression is produced which travels along the tube, as shown in the figure. This wave of compression is similar in every detail to the complete wave of compression which is produced in a steel rod by striking the end of the rod with a hammer. If the piston P is moved at a small velocity v in an outward direction and then stopped, a complete wave of rarefaction is produced.

An interesting phenomenon of wave motion occurs when a wide open hydrant is suddenly closed. The moving water is suddenly brought to rest against the valve and raised to a very high pressure, and a *wave of arrest*, *WW*, Fig. 171, travels along

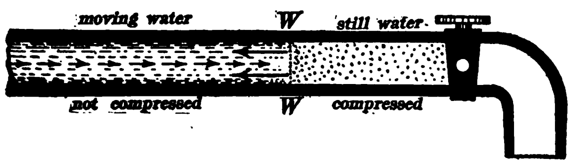


Fig. 171.

the supply pipe, as shown. At the point where the small supply pipe widens out into a large street main, the action is very similar to the action which takes place at the free end of a steel rod, as described in Art. 136. Therefore when the *wave of arrest*, *WW*, Fig. 171, reaches the street main, the uniformly compressed water in the supply pipe starts the water moving backwards into the main, this motion being established by a *wave of starting* which travels from the street main to the hydrant. When this *wave of starting* reaches the hydrant, the water in the supply pipe is moving towards the street main at a uniform velocity. This backward movement of the water is stopped immediately in the neighborhood of the valve, producing a great decrease of pressure there, and a *wave of arrest* again travels from this valve to the water main. When this *wave of arrest* reaches the water main, the water in the supply pipe is uniformly expanded, and a *wave of starting* then travels from the street main to the valve,

setting the water in motion towards the valve, as at first. This motion of the water is suddenly stopped by the valve, causing a sudden rise of pressure, as at first. This action is often repeated five or six times when a hydrant is closed, producing five or six sharp clicks in succession as the water is repeatedly brought to rest against the closed valve of the hydrant.

138. Wave pulses along wires and strings. — Imagine a long stretched wire, AB, Fig. 172, to be moving sidewise at a small

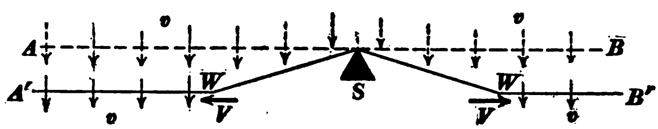


Fig. 172.

velocity, v, and to strike a rigid support or bridge, S. The portion of the wire immediately contiguous to S is stopped at once and two *waves of arrest*, WW, travel along the wire at a definite velocity V, as indicated in the figure. The portion of the wire to the left of W in Fig. 173 continues to move sidewise as if nothing

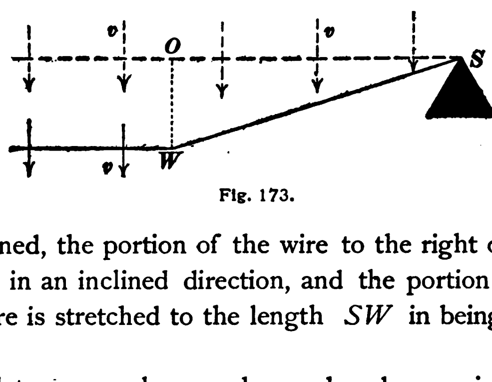


Fig. 173.

had happened, the portion of the wire to the right of W in Fig. 173 is left in an inclined direction, and the portion OS of the moving wire is stretched to the length SW in being brought to rest.

A complete wave pulse may be produced on a wire as follows: Imagine a wire to be stationary in the position $A'B'$, Fig. 172, and imagine a hammer S to come upwards at velocity v, touch the wire and continue to move at velocity v for a short time, and then rebound, leaving the wire free. The result of the first impact of S against the wire is to set up two *waves of starting*,

WW (these are *waves of arrest* when we imagine the wire to move and S to be stationary). Then when the hammer S rebounds, the tension of the inclined portions of the wire brings the wire at S to rest at once, and two *waves of arrest*, $W' W'$, travel outwards from S, as indicated in Fig. 174.

It is especially interesting to consider the details of motion of a guitar string which is pulled to one side, by placing the finger

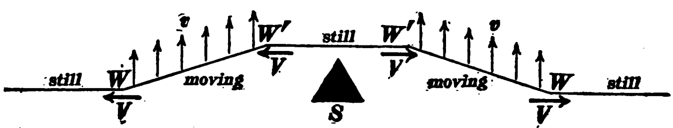


Fig. 174.

at the middle of the string, and then released. The form of the string at the instant of release is shown by A_1B_1, Fig. 175. The tension of the string sets the point p in motion at the instant of release, and two *waves of starting*, *WW*, travel to the ends of the string as shown by the diagram, A_2B_2. When these *waves of starting* reach the ends of the string, the whole wire is moving sidewise at uniform velocity v, and *waves of arrest*, $W' W'$, travel from the ends of the string towards the middle as shown in the sketch A_4B_4. When these *uaves of arrest* reach the middle of the string the entire string is momentarily stationary in the configuration shown by the sketch, A_5B_5. Then the previous action is repeated in reverse order. If the string is stiff, if the end supports A and B are not rigid (of course they are not in a guitar), and if the air friction is considerable, this extremely simple oscillatory motion of the string is greatly modified as the string continues to vibrate.

The motion of a guitar string which is plucked near one end is represented in Fig. 176. The uniformly moving straight portion of the string is equally inclined to the two stationary parts of the string.

139. Maximum strain in suddenly loaded structures. When a load is slowly applied to a structure, the strain increases

slowly until the structure is in equilibrium under the load. This value of the strain is called the *equilibrium strain*, or the *static*

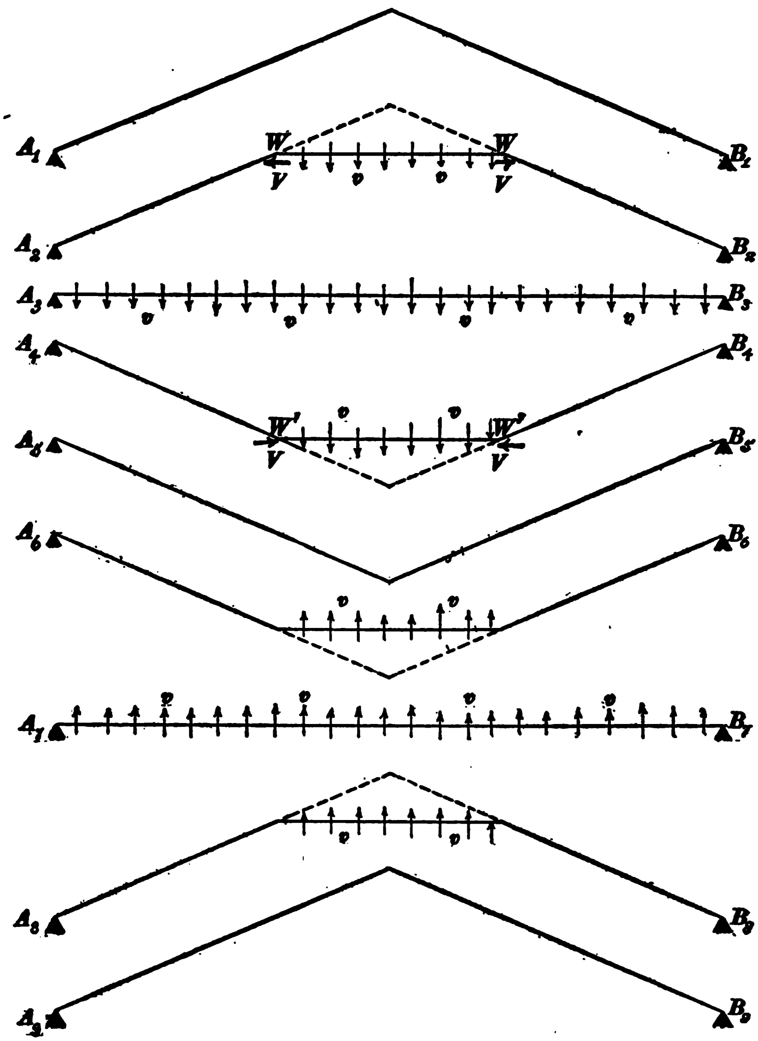


Fig. 175.

strain, corresponding to the load. When a load is suddenly applied to a structure, the momentum which the structure gains

while the strain is increasing to the equilibrium value carries the strain beyond the equilibrium value. *The maximum strain pro-*

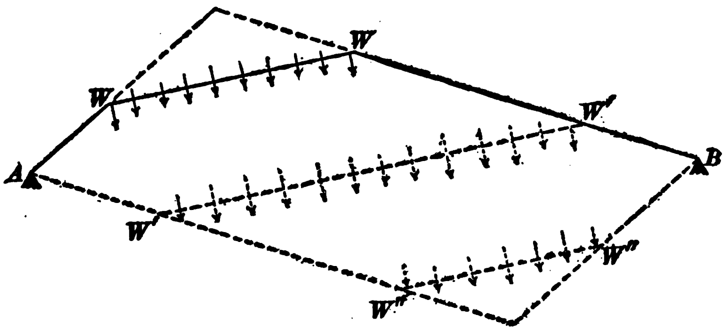


Fig. 176.

duced by a suddenly applied load is equal to two times the equilibrium strain corresponding to the load, friction being negligible.

Case 1. When the mass of the elastic structure is negligible in comparison with the mass of the load. Figure 177 represents a helical spring in its unstretched initial position, and a weight attached to the spring, but supported by the hand. When the weight is released, it oscillates up and down, coming back to its initial position, or nearly to its initial position, repeatedly. But the equilibrium position of an oscillating body is midway between its extreme positions, and therefore the lowest position reached by the oscillating weight in Fig. 177 is at a distance, $2x$, below the initial position, where x is the distance between the equilibrium position and the initial position, as shown.

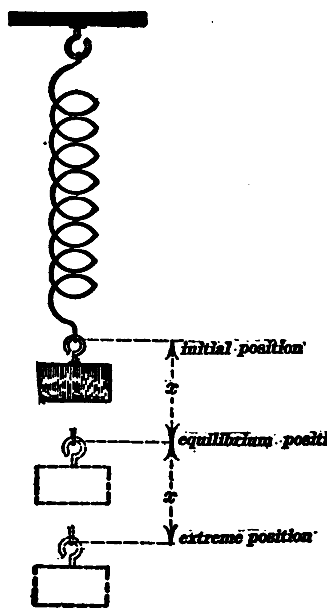


Fig. 177.

Case 2. When the mass of the elastic structure is not negligible.* A force, F, is suddenly applied to the top of a steel column, as shown in Figs. 178 and 179. The top of the column is immediately compressed by the full value of the force (equilibrium strain), and also set moving downwards at a definite velocity, v. This equilibrium compression and the downward velocity, v, are imparted to the successive portions of the column by the *wave of starting*, W, which travels downwards, as indicated in

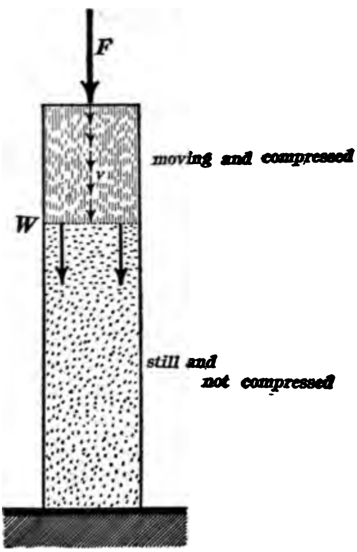


Fig. 178.

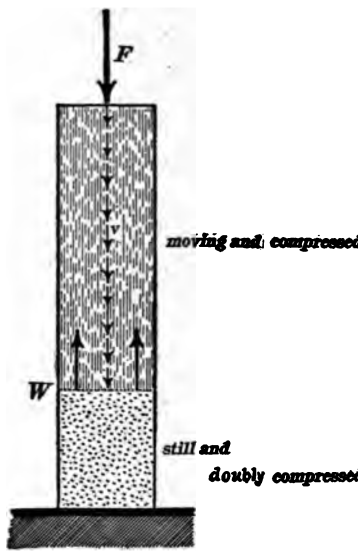


Fig. 179.

Fig. 178. At the instant when this wave reaches the bottom of the column, the column is everywhere compressed to the equilibrium value; but its downward motion creates a double compression at the bottom, and this double compression extends upwards with the upward movement of the *wave of arrest*, W, Fig. 179. When this *wave of arrest* reaches the top of the column, the entire column is under a strain twice as great as the equilibrium

* In the discussion of the case here given, the load is assumed to have a negligible mass. Substantially the same conclusion would be reached, if the mass of the load were taken into account, but the argument would be very complicated.

strain corresponding to the load. The subsequent oscillations of the column need not be considered.

140. Longitudinal waves and transverse waves. — The kind of water waves described in Art. 135, and the waves described in Arts. 136 and 137 are called *longitudinal waves* because the actual motion of the water or steel or air in the wave is parallel to the direction of progression of the wave, v parallel to V. Sound waves in the air are longitudinal waves. The kind of waves described in Art. 138 are called *transverse waves* because the actual motion of the wire in the wave (between W and W' in Fig. 174) is at right angles to the direction of progression of the wave, v at right angles to V. Light waves in the ether are transverse waves. In ordinary water waves, each particle of water describes a circular or elliptical orbit, the plane of which is vertical and parallel to the direction of progression of the waves, that is to say, ordinary water waves are neither purely longitudinal nor purely transverse waves.

Polarization of transverse waves. — Imagine the end of a long stretched rope to be moved rapidly up and down so as to produce waves traveling along the rope. The direction of oscillation of the rope is everywhere in a vertical plane; transverse waves in which the direction of oscillation is parallel to a certain plane are called plane *polarized waves.*

141. The wave front. — The idea of the wave front plays a very important part in the wave theory of light. Imagine a disturbance to take place at a point on the surface of a pond. If this disturbance is simple in character, a sharply defined wave passes out from it. If the disturbance is complicated, for example, if a handful of pebbles is thrown into the water, the waves in the immediate neighborhood of the disturbance are utterly confused. At a great distance from a disturbance, however, the waves always become sharply defined, whether the disturbance be simple or complicated, that is to say, the waves form definite ridges on the surface of the water, and a line can be drawn along the water's surface so as to pass through points where the water

surface is in a similar state of motion and a similar condition of distortion. Such a line is called a *wave front.* In the case of sound and light waves, the wave front is a surface imagined to be drawn through the air or ether, the motion and distortion of the medium being the same at every point of this surface. A wave which has a plane front is called a *plane wave* and a wave which has a spherical front is called a *spherical wave.* The importance of the idea of wave front in optics is due to the fact that the direction of progression of the wave is at right angles to the wave front when the medium is isotropic.

142. Free oscillations of elastic systems. — The discussion given in Arts. 136 and 138 illustrate the application of the wave theory to the study of oscillatory motion. A complete development of this aspect of the wave theory depends upon a consideration of wave trains, and this development is usually included in treatises on acoustics.

Simple modes of oscillation. — An elastic system, such as a stretched string, an air column, a steel rod, or a bell, may oscillate in such a way that every particle of the system performs *simple harmonic motion of the same frequency.* When a system vibrates in this way there are, in general, certain places called *nodes* where there is no motion at all, and the vibrating regions between these nodes are called *vibrating segments.* A system vibrating in this way is said to vibrate in a *simple mode.* A given elastic system may perform a great variety of simple modes of oscillation. For example, a string may vibrate as a whole, or in two parts, or in three parts, or in any number of parts. A column of air in a tube may vibrate as a whole, or in halves, or in any number of parts; the notes of a bugle, for example, are produced by causing the air column in the instrument to perform different simple modes of oscillation. A flat metal plate clamped at its center may be set vibrating by drawing a violin bow across its edge. If sand be strewn upon the plate, the sand is thrown away from the vibrating segments and heaped up on the nodal lines, forming very beautiful figures, which were first studied by

Chladni. Figure 180 shows some of the sand figures obtained by Chladni and depicted in his treatise on Acoustics (Leipzig, 1787).

In a vibrating string, *nodal points* separate the vibrating segments; in a vibrating plate, *nodal lines* separate the vibrating segments; and in a system of three dimensions, *nodal surfaces* separate the vibrating segments.

Compound modes of oscillation. — An elastic system may perform any number of its simple modes of oscillation simultaneously. In this case, the system is said to oscillate in a *compound mode* and each particle of the system performs simultaneously the various simple harmonic movements corresponding to the respective simple modes of oscillation that are present in the oscillatory motion of the system. For example, a metal plate, struck by a hammer, performs simultaneously a great number of the simple modes of oscillation of which it is capable, and usually, the musical tone corresponding to each simple mode of oscillation can be separately heard. The simple type of oscillation of a steel rod, described in Art. 136, and the simple type of oscillation of a string described in Art. 138, constitute compound modes of oscillation in the sense in which this term is here used, and, as a matter of fact, it is possible to detect a series of distinct tones in the sound that is emitted by a steel rod or string vibrating in the manner described in Arts. 136 and 138.

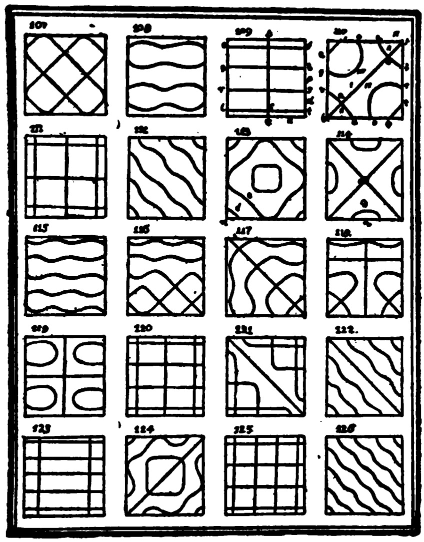


Fig. 180.

143. Forced oscillations of elastic systems. Resonance.—When an elastic system like a string or a steel rod, is distorted and released, it performs *free oscillations* at a definite frequency. When an elastic system is acted upon by a periodic force from outside, the system is made to oscillate at the same frequency as the applied force. Such oscillations are called *forced oscillations.*

Consider an oscillating elastic system. If the oscillations are free, then the forces required to overcome the inertia of the moving parts of the system as they are repeatedly stopped and started, will be supplied wholly by the elastic reactions in the system, and the forces required to produce the repeated distortion of the system will be supplied by the inertia reactions in the system, so that the only outside force that would be required to maintain free oscillations of a system is a force sufficient to overcome friction, and this force must be of the same frequency as the free oscillations of the system. If, however, the outside force is not of the same frequency as the free oscillations of the system, a large part of the force must be used to overcome the inertia reactions or the elastic reactions in the system, and only a small portion of the force is available for overcoming friction. Therefore *a periodic force acting on an elastic system produces a maximum of violence of oscillation if the frequency of the force is the same as the frequency of the free oscillations of the system.* This phenomenon is called *resonance.* It is exemplified by the very decided oscillatory motion of a bridge, which is produced by a comparatively weak force of the proper frequency. Thus, the rhythmic motion of a trotting horse sets a bridge into violent oscillatory motion if the rhythm of the movements of the horse is the same as the rhythm of free oscillation of the bridge. It usually requires a long continued action of a periodic force to fully establish the comparatively violent oscillations of a system by resonance.

PROBLEMS.

168. What is the most suitable velocity of propulsion of a canal boat in a canal 7 feet deep?

Note.—When a canal boat is propelled at the velocity of wave propagation in a canal, it rides on the wave which it produces, and less force is required to propel it than would be required if the boat moved at a considerably less velocity; also, the agitation of the water is less pronounced than it would be at considerably less velocity, and the washing of the canal banks is correspondingly less.

169. A helical spring 3 feet long weighs one-half pound and it is elongated one inch by the force of 2 pounds-weight. One end of the spring is attached to a support and the spring is stretched and suddenly released. How many complete oscillations does it make per second?

170. Plot a curve of which abscissas represent elapsed time and of which the ordinates represent the varying distance of the free end of the helical spring of problem 169 from its fixed end; the initial stretch of the spring being $\frac{3}{4}$ of an inch.

171. Plot a curve of which the abscissas represent elapsed time and of which the ordinates represent the varying distance of the middle point of the helical spring of problem 169 from its fixed end; the initial stretch being $\frac{3}{4}$ of an inch.

172. A steel rod 6 feet long is supported at its center, as shown in Fig. 172*p*, and one end of the rod is stroked with a rosined

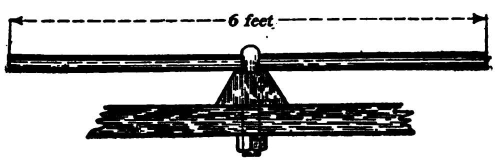


Fig. 172*p*.

cloth, causing the rod to vibrate lengthwise. Required the number of vibrations made by the rod per second.

Note.—When a rod is supported as indicated in the figure, each end of it oscillates in the manner represented in Figs. 155 to 158. The density of the steel is 7.78 times $62\frac{1}{2}$ pounds per cubic foot and its stretch modulus is 30,000,000 pounds per square inch.

173. A brass rod 11.63 feet long is mounted as indicated in Fig. 172*p*, and when stroked with a rosined cloth it gives a musical tone in unison with a standard tuning fork which has a frequency of 512 complete vibrations per second. Calculate the stretch

modulus of brass, its density being 8.4 times $62\frac{1}{2}$ pounds per cubic foot.

174. Assuming that the face of a hammer comes squarely against the end of a steel rod, calculate the greatest velocity of the hammer which will not batter the end of the rod. The resilience of the steel of the rod is 8,640 foot-pounds per cubic foot.

Note.—An endwise hammer blow on a steel rod produces a wave of compression of which the velocity-phase is equal to the velocity of the hammer, as explained in Art. 136. The compression which exists in this wave is a compression of which the potential energy per unit volume is equal to the kinetic energy per unit volume due to the motion of the steel. The battering of the end of the rod is due to the compression of the steel beyond the elastic limit.

175. The velocity of an air wave along an air pipe is 1,000 feet per second. The air in the pipe is slightly compressed, and then one end of the pipe is suddenly opened. The pipe is 25 feet long. How many complete oscillations does the air in the pipe make per second?

Note. An air wave is reflected from the open end of a pipe in the same way that a wave of compression or stretch is reflected from the free end of a steel rod.

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